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Integrable Hierarchies and Modern Physical Theories

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CONTENTS

Symmetry Flows, Conservation Laws and Dressing Approach to the Integrable Models[1]

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Abstract

The graded affine Lie algebras provide a framework in which the dressing method is applied to the generic type of integrable models. The dressing formalism is used to develop a unified approach to various symmetry flows encountered among the integrable hierarchies and to describe related conservation laws.

1 Introduction

We consider the integrable hierarchies of differential equations in the framework of a linear spectral eigenvalue equation defined in terms of the matrix Lax operator. In order to develop this construction of integrable models, it is, at the bare minimum, necessary to define a notion of affine algebra with a graded structure together with a semisimple element E of this algebra with a fixed positive grade (this grade will be chosen for simplicity to one). Such an algebraic approach was coined a generalized Drinfeld-Sokolov method in a series of papers [1, 2], which extended early results of Drinfeld, Sokolov and Wilson [3, 4]. By including various non-standard gradations, the scheme was shown to encompass a class of the constrained KP hierarchies [5]. The simplicity and solvability of the integrable model in the algebraic formalism ultimately springs from the fact that its building block-the Lax operator-can be rotated into much simpler abelian Lax operator by employing the so-called dressing transformation. Supplementing the algebraic method by the dressing transformation gives rise to a simple and elegant construction of all symmetry flows of the integrable models, including the isospectral deformations of the underlying Lax matrices. The conservation laws follow easily and, via cohomological arguments, find a transparent interpretation in terms of the tau-function. The scheme involves only the dressing transformation operators in associating a symmetry flow to every algebra element which commutes with a semisimple element E. For some gradations such algebra elements constitute a non-abelian sub-algebra

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and accordingly the corresponding symmetry flows span a non-abelian algebra. The method extends easily to incorporate the Virasoro symmetry of the integrable models. Choosing the underlying algebra to be a super-algebra results in the fermionic symmetry flows.

This purely algebraic approach is shown to reconcile straightforwardly with the framework of additional symmetries obtained within the calculus of pseudo-differential operators [6, 7].

The significant portion of our presentation should not come as a surprise to experts in the field, see f.i. papers [1, 2, 8, 9, 10, 11]. What we believe has a claim to novelty in this presentation is a unified approach to the subject of symmetry flows which naturally includes the non-abelian flows and their connection to the tau-functions. We also point out a link to an alternative pseudo-differential approach to the symmetries of the Lax hierarchies.

2 Dressing Technique

Let $\widehat{\mathcal{G}}$ be the affine Lie algebra with an integral gradation: $\widehat{\mathcal{G}} = \oplus_{n\in\mathbb{Z}}\, \widehat{\mathcal{G}}_n$ with respect to the grading operator $Q_{\mathbf{s}}$ such that $[Q_{\mathbf{s}}\,,\, \widehat{\mathcal{G}}_n] = n\, \widehat{\mathcal{G}}_n$. For the background material on gradation in the framework of the affine Lie algebras see Section [6].

We consider the matrix Lax operator

$$L = D_x + E + A \tag{2.1}$$

which as will be shown below defines an integrable hierarchy associated to the linear spectral problem :

$$L\Psi_0 = (\partial_x + E + A)\,\Psi_0 = 0 \tag{2.2}$$

For the class of models with $\widehat{\mathcal{G}} = \widehat{sl}(M+K+1)$ the corresponding choice of elements E and A is described in Section [6] together with other basic information. The semisimple element E defines a direct sum decomposition of the algebra $\widehat{\mathcal{G}}$:

$$\widehat{\mathcal{G}} = \mathcal{K} \oplus \mathcal{M}\,, \tag{2.3}$$

where $\mathcal{M} = \mathrm{Im}\ (\mathrm{ad}\ E)$ and $\mathcal{K}$ is a centralizer of E:

$$\mathcal{K} = \mathrm{Ker}\ (\mathrm{ad}\ E) \equiv \{x \in \widehat{\mathcal{G}} |\, [\, x\,,\, E\,] = 0\}\,. \tag{2.4}$$

From Jacobi identities we find the algebraic relations:

$$[\,\mathcal{K}\,,\mathcal{K}\,] \subset \mathcal{K} \quad ; \quad [\,\mathcal{K}\,,\mathcal{M}\,] \subset \mathcal{M} \tag{2.5}$$

For simplicity of presentation we work in this paper with E of grade one only ($E \in \widehat{\mathcal{G}}_1$). $\mathcal{K}$ is a graded sub-algebra of $\widehat{\mathcal{G}}$, i.e. $\mathcal{K} = \oplus_{n\in\mathbb{Z}}\, \mathcal{K}_n$. We will allow here gradations which are more general than the principal gradation. For that reason we do not assume that $\mathcal{K}$ itself is abelian and therefore in general it differs from its (abelian) center defined as :

$$\mathcal{C}(\mathcal{K}) \equiv \{x \in \mathcal{K} | [x, y] = 0\,,\, \forall y \in \mathcal{K}\}\,. \tag{2.6}$$

The potential A in L is chosen to belong to the grade zero component of $\mathcal{M}$ ($\mathcal{M}_0$) in the grade zero component of $\widehat{\mathcal{G}}$ ($\widehat{\mathcal{G}}_0$).

Based on the above setup one can show that the Lax operator L can be gauge-rotated into Ker (ad E) by a dressing transformation given by $\mathrm{Ad}(U^{-1})$:

$$L = D_x + E + A \;\; \rightarrow \;\; L_K = U^{-1}\, L\, U = D_x + E + K^- \tag{2.7}$$

$$K^- \;\; \equiv \;\; \sum_{j=1}^{\infty} K^{(-j)} \in \mathcal{K}_- \tag{2.8}$$

where $\mathcal{K}_- = \oplus_{j=-1}^{-\infty} \mathcal{K}_j$ is a negative part of $\mathcal{K}$ w.r.t. to the given grading. Let U be an exponentiation of negative grade generators, $U = e^{\mathfrak{u}}$, with $\mathfrak{u} = \sum_{j=1}^{\infty} u^{(-j)}$ and $u^{(-j)} \in \mathcal{M}_{-j}$. Note, the absence of the grade zero components in K^-. This follows from the fact that the projection of $U^{-1}\, L\, U$ on grade zero is given by $\left[\, E\, ,\, u^{(-1)}\,\right] + A$ and lies entirely in Im (ad E). This expression can be put to zero by appropriately choosing the $\mathcal{M}$ component $u^{(-1)}$ as a local function of A. From identities

$$\left(U^{-1} E U\right)_{-j} \;\; = \;\; \sum_{r=1}^{j+1} (-1)^r \frac{1}{r!} \sum_{k_i : \sum k_i = j+1} [u^{(-k_1)}, [u^{(-k_2)}, \ldots [u^{(-k_r)}, E] \ldots]$$

$$\left(U^{-1} \partial_x U\right)_{-j} \;\; = \;\; \sum_{r=1}^{j} (-1)^{r-1} \frac{1}{r!} \sum_{k_i : \sum k_i = j} [u^{(-k_1)}, \ldots [u^{(-k_{r-1})}, \partial_x u^{(-k_r)}] \ldots]$$

we find that the generic expression at the grade $-j$ in (2.7) must be of the form $\partial_x u^{(-j)} + \left[\, E\, ,\, u^{(-j-1)}\,\right] + \left[\, A\, ,\, u^{(-j)}\,\right] + \ldots = K^{(-j)}$ where omitted terms contain only $u^{(-l)}, l < j$. Due to the fact that $K^- \in \mathcal{K}$, this recursion procedure allows the choice of $u^{(-j-1)}|_{\mathcal{K}} = 0$. The remaining $\mathcal{M}$-component $u^{(-j-1)}$ is given in terms of previously known elements $u^{(-l)}, l \leq j$. Note, that all elements $u^{(-j)}$ besides belonging to $\mathcal{M}$, are local expressions of polynomials of components of A.

Next, we proceed to "gauge away" the term K^- in (2.7) using that, according to relation (2.5), $\mathcal{K}$ is a sub-algebra of $\widehat{\mathcal{G}}$. Following the standard arguments, the dressing of the Lax operator L_K proceeds according to :

$$S^{-1}\, L_K\, S = D_x + E \tag{2.9}$$

where $S = e^{\mathfrak{s}}$ is an exponentiation of negative grade generators from $\mathcal{K}$, so that $\mathfrak{s} = \sum_{j=1}^{\infty} s^{(-j)} \in \mathcal{K}$. Indeed, contribution to $S^{-1}\, L_K\, S$ at grade -1 is $\partial_x s^{(-1)} + K^{(-1)}$, which determines $s^{(-1)}$. At grade -2 we find $\partial_x s^{(-2)} + K^{(-2)} + \left[\, K^{(-1)}\, ,\, s^{(-1)}\,\right] + \frac{1}{2}\left[\, \partial_x s^{(-1)}\, ,\, s^{(-1)}\,\right]$, which can be put to zero by the appropriate choice of $s^{(-2)}$. This process can be continued recursively. For abelian $\mathcal{K}$ it yields $K^- = -S^{-1}\partial_x S$. Note, that in contrast to exponential U in (2.7), the exponential S in (2.9) has a non-local character.

Combining results of eqs.(2.7) and (2.9) we arrive at the solution to the problem :

$$\Theta^{-1}\, (D + E + A)\, \Theta = D + E \tag{2.10}$$

with transformation $\Theta \equiv US$ given by expansion in the terms of negative grading such that $\Theta = \exp\left(\sum_{l<0} \theta^{(l)}\right) = 1 + \theta^{(-1)} + \ldots$.

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Let now b_n be in the center $\mathcal{C}_n(\mathcal{K})$ of $\mathcal{K}_n$ and Θ the dressing operator satisfying eq.(2.10). We associate to $\Theta b_n \Theta^{-1}$ the following descending expansion in grading:

$$\Theta b_n \Theta^{-1} = b_n + \sum_{k=0}^{\infty} \beta_n^{(n-k-1)} \tag{2.11}$$

where the term $\beta_n^{(n-k-1)}$ has a $(n-k-1)$-grade, i.e. $\left[\, Q_{\mathbf{s}}\, ,\, \beta_n^{(n-k-1)}\,\right] = (n-k-1)\, \beta_n^{(n-k-1)}$ with respect to the grading operator $Q_{\mathbf{s}}$ (see f.i. (6.3)). Next, we define :

$$B_n = \left(\Theta b_n \Theta^{-1}\right)_+ = b_n + \sum_{k=0}^{n-1} \beta_n^{(n-k-1)} \tag{2.12}$$

Note, that since b_n commutes with S, the right hand side of eq. (2.11) can be rewritten in an explicitly local form $\Theta b_N \Theta^{-1} = U b_N U^{-1}$. Also, note that $B_1 = (UEU^{-1})_+ = E + A$.

3 Symmetry Flows and Conservation Laws

3.1 Symmetry Flows and Isospectral Times

In this Section we show how to associate a symmetry flow δ_X to any constant element $X \in \mathcal{K} = \mathrm{Ker}$ (ad E) of a positive grade.

Definition 3.1 *For a constant element X_m, with grade $m > 0$, belonging to $\mathcal{K}_m$ we define a resolvent of X_m as :*

$$X_m^{\Theta} \equiv Ad(\Theta) X_m = \Theta X_m \Theta^{-1} \tag{3.1}$$

Next, apply $\mathrm{Ad}\,(\Theta)$ on the bracket:

$$[\, X_m\, ,\, D_x + E\,] = 0 \tag{3.2}$$

to obtain:

$$[L\, ,\, X_m^{\Theta}] = 0 \tag{3.3}$$

Definition 3.2 *Let X_m be in $\mathcal{K}_m$. Define a transformation δ_{X_m} associated to X_m by*

$$\delta_{X_m}\Theta = (\Theta X_m \Theta^{-1})_- \Theta \rightarrow \delta_{X_m}\Theta \equiv (X_m^{\Theta})_- \Theta \quad , \quad m \geq 0 \tag{3.4}$$

To b_n in the center $\mathcal{C}_n(\mathcal{K})$ of $\mathcal{K}_n$ we associate a flow $\delta_{b_n} \equiv d/dt_n$:

$$\delta_{b_n}\Theta = \frac{d}{dt_n}\Theta = (\Theta b_n \Theta^{-1})_- \Theta = \Theta b_n - B_n \Theta \tag{3.5}$$

and similarly

$$\frac{d}{dt_n}\Theta^{-1} = -b_n\,\Theta^{-1} + \Theta^{-1}\,B_n \tag{3.6}$$

Note, that eq. (2.10) is equivalent to $\partial_x\Theta = \Theta E - (E+A)\Theta$ and since $B_1 = E + A$ the definitions (2.10) and (3.5) imply that $d/dt_1 = \partial_x$.

Note also, that according to the definition (3.5), B_n can be rewritten as

$$B_n = \Theta b_n \Theta^{-1} + \Theta\left(\frac{d}{dt_n}\Theta^{-1}\right) \tag{3.7}$$

The action of the transformation δ_{X_m} applied on the potential A is described by the following Lemma.

Lemma 3.1 *Let $X_m \in \mathcal{K}_m$, then*

$$\delta_{X_m} A = [L\,,\,(X_m^{\Theta})_+] \tag{3.8}$$

Proof. The proof of (3.8) goes as follows. First from eq.(3.4) we find

$$0 = \delta_{X_m}\left(\Theta^{-1}\,L\,\Theta\right) = -[\Theta^{-1}\delta_{X_m}\Theta\,,\,\Theta^{-1}\,L\,\Theta] + \Theta^{-1}\,(\delta_{X_m}L)\,\Theta \tag{3.9}$$

which for $\delta_{X_m}L = \delta_{X_m}A$ gives

$$\delta_{X_m}A = \Theta[\Theta^{-1}\delta_{X_m}\Theta\,,\,\Theta^{-1}\,L\,\Theta]\Theta^{-1} = [\delta_{X_m}\Theta\,\Theta^{-1}\,,\,L] \tag{3.10}$$

Eq. (3.8) follows now by virtue of the resolvent identity (3.3) and definition (3.2). □

Consider, again a constant element b_n, of grade n $(n>0)$ such that $b_n \in \mathcal{C}_n(\mathcal{K})$. From the above considerations one gets

$$\frac{dA}{dt_n} = [L\,,\,B_n] \tag{3.11}$$

The commutative symmetry transformations in (3.5) and (3.11) are called *the isospectral flows*. Note, that the identity $U\,b_N\,U^{-1} = b_N^{\Theta}$ ensures locality of the corresponding conservation laws.

It is natural to generalize the dressing relation (2.10) to other isopectral flows by defining

$$L_N \equiv \Theta\left(\frac{d}{dt_N} + b_N\right)\Theta^{-1} = \frac{d}{dt_N} - \left(\frac{d}{dt_N}\Theta\right)\Theta^{-1} + U b_N U^{-1} \tag{3.12}$$

which coincides with (2.10) for $N=1$. From (3.7) we find

$$L_N = \frac{d}{dt_N} + B_N = \Psi_0 \frac{d}{dt_N}\Psi_0^{-1} \tag{3.13}$$

for the wave-function Ψ_0 :

$$\Psi_0 \equiv \Theta \exp\left(-\sum_{N=1}^{\infty} b_N t_N\right) \qquad b_1 = E \quad ; \quad t_1 = x \tag{3.14}$$

defined in terms of the parameters t_N such that $[\,d/dt_{N'}\,,\,t_N\,] = \delta_{N'N}$. Due to (3.12) such wave-function Ψ_0 satisfies :

$$L_N\Psi_0 = 0 \qquad N = 1,2,\ldots \tag{3.15}$$

and therefore is a solution for the underlying linear spectral problem (2.2). Eqs.(3.15) are equivalent to an hierarchy of evolution equations :

$$\frac{d\Psi_0}{dt_N} = -B_N\Psi_0 \tag{3.16}$$

Moreover, by conjugating with $\mathrm{Ad}(\Theta)$ identities:

$$\left[\frac{d}{dt_N} + b_N\,,\,D_x + E\right] = 0 \tag{3.17}$$

$$\left[\frac{d}{dt_N} + b_N\,,\,\frac{d}{dt_M} + b_M\right] = 0 \tag{3.18}$$

we obtain the Zakharov-Shabat equations

$$[\,L_N\,,\,L\,] = \frac{dA}{dt_N} - \partial_x B_N + [\,B_N\,,\,E+A\,] = 0 \tag{3.19}$$

$$[\,L_N\,,\,L_M\,] = \frac{dB_M}{dt_N} - \frac{dB_N}{dt_M} + [\,B_N\,,\,B_M\,] = 0 \tag{3.20}$$

We recognize in these equations compatibility conditions for the linear relations (2.2) and (3.15).

We will now use (3.8) to show that eq. (3.4) for arbitrary $X_m \in \mathcal{K}$ generates a well-defined symmetry transformation of the above model, meaning that

- $\delta_{X_m}A \in \mathcal{M}_0$
- transformations δ_{X_m} commute with the isospectral flows.
- transformations δ_{X_m} close into an algebra.

In other words we have the following Lemma

Lemma 3.2 *The transformations in (3.4) (or in (3.8)) are symmetry transformations of the model defined by $L\Psi_0 = 0$ and $dL/dt_N = [L\,,\,B_N]$.*

Proof. The proof goes as follows. First notice that (3.3) implies:

$$[L\,,\,(X_m^{\Theta})_+] = -[L\,,\,(X_m^{\Theta})_-] \tag{3.21}$$

The conventional "dressing" argument compares grades on both sides of equation (3.21). The left hand side involves terms with grades ≥ 0 while grades on the right hand side are between 0 and $-\infty$. Consequently, each side of eq.(3.21) lies in the zero grade sub-algebra. Correspondingly, the contributions of the above terms are equal to:

$$\delta_{X_m}A = [D_x + A\,,\,(X_m^{\Theta})_0] = -[E\,,\,(X_m^{\Theta})_{-1}] \tag{3.22}$$

The last equality ensures that $\delta_{X_m} A$ is in $\mathcal{M}_0$ and therefore the transformation generated by (3.8) or (3.10) is well-defined.

To complete the proof of Lemma (3.2) we will show that the algebra of transformations from (3.4) closes and commutes with the isospectral flows. Let us first discuss the algebra closure. Consider:

$$(\delta_{X_m}\delta_{X_n} - \delta_{X_n}\delta_{X_m})\,\Theta = \delta_{X_m}\left((X_n^\Theta)_-\Theta\right) - \delta_{X_n}\left((X_m^\Theta)_-\Theta\right) \tag{3.23}$$

To proceed we need to find $\delta_{X_n}(X_m^\Theta)_-$.

$$\begin{aligned}\delta_{X_n}\left(\Theta X_m \Theta^{-1}\right)_- &= [(\delta_{X_n}\Theta)\Theta^{-1}\,,\,\Theta\, X_m\,\Theta^{-1}]_- = [(X_n^\Theta)_-\,,\,X_m^\Theta]_- \\ &= [(X_n^\Theta)_-\,,\,(X_m^\Theta)_-] + [(X_n^\Theta)_-\,,\,(X_m^\Theta)_+]_- \end{aligned}\tag{3.24}$$

Inserting these results into (3.23) we get

$$(\delta_{X_m}\delta_{X_n} - \delta_{X_n}\delta_{X_m})\,\Theta = f^k_{mn}(X_k^\Theta)_-\Theta = f^k_{mn}\delta_{X_k}\Theta = \delta_{[X_m\,,\,X_n]}\Theta \tag{3.25}$$

after comparing with the algebra of generators in $\mathcal{K}$

$$[\,X_m\,,\,X_n\,] = f^k_{mn}X_k \tag{3.26}$$

and noticing that the left hand side after dressing by Θ and projecting on the negative modes becomes :

$$\left[\,X_m^\Theta\,,\,X_n^\Theta\,\right]_- = [(X_m^\Theta)_-\,,\,(X_n^\Theta)_-] + [(X_m^\Theta)_-\,,\,(X_n^\Theta)_+]_- + [(X_m^\Theta)_+\,,\,(X_n^\Theta)_-]_-$$

Now, consider commutation with the isospectral flows. Since $b_N \in \mathcal{C}(\mathcal{K})$ we have

$$[\,X_m\,,\,b_N\,] = 0 \tag{3.27}$$

and the same arguments as above yield this time:

$$\left(\delta_{X_m}\frac{d}{dt_N} - \frac{d}{dt_N}\delta_{X_m}\right)\Theta = 0 \tag{3.28}$$

□

3.2 Conservation Laws

We now associate to each $X_n \in \mathcal{K}$ the following class of objects.

Definition 3.3 *Define maps $\mathcal{J}$, Ω: $\mathcal{K} \to \mathbb{C}$ as :*

$$\mathcal{J}(X_n) \equiv \mathrm{Tr}\left([\,Q_{\mathbf{s}}\,,\,\Theta\,]\;X_n\Theta^{-1}\right) \tag{3.29}$$

$$\Omega(X_n) \equiv -\,\mathrm{Tr}\left(EX_n^\Theta\right) = -\,\mathrm{Tr}\left(E\Theta X_n\Theta^{-1}\right) \tag{3.30}$$

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Here, $\mathrm{Tr}(\ldots) = \mathrm{tr}(\ldots)_0$ involves both the conventional matrix trace operation tr as well as a projection on the zero grade.

The above objects are related through :

Proposition 3.1

$$\partial_x\mathcal{J}(X_n) = \Omega(X_n) \tag{3.31}$$

Proof. The proof follows by taking $m = 1$ in eqs. (3.5) and (3.6) which produces:

$$\partial_x\mathcal{J}(X_n) = -\,\mathrm{Tr}\left([\,Q_{\mathbf{s}}\,,\,B_1\,]\;\Theta\, X_n\,\Theta^{-1}\right) \tag{3.32}$$

Since B_1 is equal to $E + A$ and $[\,Q_{\mathbf{s}}\,,\,E + A\,] = E$ we get from eq. (3.32). $\partial_x\mathcal{J}(X_n) = -\,\mathrm{Tr}\,(E\Theta\, X_n\,\Theta^{-1}\,) = \Omega(X_n)$ □

Proposition 3.2

$$\delta_{X_n}\mathcal{J}(X_m) - \delta_{X_m}\mathcal{J}(X_n) = f^k_{nm}\mathcal{J}(X_k) \tag{3.33}$$

where f^k_{mn} is the structure constant of the sub-algebra $\mathcal{K}$ from relation (3.26).

Proof. We make use of the property: $\mathrm{Tr}\,(A\,[\,Q_{\mathbf{s}}\,,\,B\,]) = -\,\mathrm{Tr}\,([\,Q_{\mathbf{s}}\,,\,A\,]\;B)$, satisfied by the trace Tr. Accordingly,

$$\delta_{X_n}\mathcal{J}(X_m) = -\,\mathrm{Tr}\left(\left(\Theta\, X_n\,\Theta^{-1}\right)_-\left[\,Q_{\mathbf{s}}\,,\,\Theta\, X_m\,\Theta^{-1}\,\right]\right) \tag{3.34}$$

and therefore

$$\delta_{X_n}\mathcal{J}(X_m) - \delta_{X_m}\mathcal{J}(X_n) = \mathrm{Tr}\left(\left(\Theta\, X_m\,\Theta^{-1}\right)\left[\,Q_{\mathbf{s}}\,,\,\left(\Theta\, X_n\,\Theta^{-1}\right)\,\right]\right) \tag{3.35}$$

which is equal to

$$\begin{aligned} &\mathrm{Tr}\left(\left(\Theta\, X_m\,\Theta^{-1}\right)[\,Q_{\mathbf{s}}\,,\,\Theta\,]\;X_n\,\Theta^{-1}\right) + \mathrm{Tr}\left(\left(\Theta\, X_m\,\Theta^{-1}\right)\Theta\, nX_n\,\Theta^{-1}\right) \\ -\;\; &\mathrm{Tr}\left(\left(\Theta\, X_m\,\Theta^{-1}\right)\Theta\, X_n\,\Theta^{-1}\,[\,Q_{\mathbf{s}}\,,\,\Theta\,]\,\Theta^{-1}\right)\end{aligned} \tag{3.36}$$

The middle term vanishes being equal to $n\,\mathrm{Tr}(X_mX_n)$ while the first and last terms combine to give :

$$\mathrm{Tr}\left([\,Q_{\mathbf{s}}\,,\,\Theta\,]\;(X_nX_m - X_mX_n)\,\Theta^{-1}\right) = \mathcal{J}\left([\,X_n\,,\,X_m\,]\right) \tag{3.37}$$

as announced in the proposition. □

Result of the Proposition 3.2 can be reformulated in the following more formal statement.

Lemma 3.3 *A one-form $\mathcal{J}(\cdot)$ on $\mathcal{K}$ with values in $\mathbb{C}$ defined in (3.3) is a closed one-cocycle:*

$$d\mathcal{J}(X_n, X_m) = 0 \tag{3.38}$$

with respect to the usual Cartan-Chevalley-Eilenberg differential d given by the formula:

$$\begin{aligned} d\mathcal{J}(X_1,\ldots,X_n) &= \sum_{i=1}^{n}(-1)^{(i-1)}\delta_{X_i}\mathcal{J}(X_1,\ldots,\hat{X}_i,\ldots,X_n) \\ &+ \sum_{j<k}(-1)^{(j+k)}\mathcal{J}\left([X_j\,,\,X_k],\ldots,\hat{X}_j,\ldots,\hat{X}_k,\ldots,X_n\right)\end{aligned} \tag{3.39}$$

As a corollary of Proposition 3.1 we obtain that $\Omega(X_n)$ too is a closed one-cocycle

$$d\Omega(X_n, X_m) = 0\,. \tag{3.40}$$

We will now address a question whether the closed cocycle $\Omega(X_n)$ is also exact, namely whether there exists Ω_0 such that

$$\Omega(X_n) = d\,\Omega_0(X_n) \tag{3.41}$$

From the definition (3.2) it follows that $\delta_{X_m}\Theta = (USX_mS^{-1}U^{-1})_{-}US$. Assuming that δ_{X_m} acts as a derivative (satisfies Leibniz rule) we get:

$$S^{-1}\delta_{X_m}S = -S^{-1}(U^{-1}\delta_{X_m}U)S + S^{-1}U^{-1}(USX_mS^{-1}U^{-1})_{-}US \tag{3.42}$$

After multiplying by E and taking the trace eq. (3.42) becomes:

$$\mathrm{Tr}\left(E\,S^{-1}\delta_{X_m}S\right) = -\,\mathrm{Tr}\left(E\,U^{-1}\delta_{X_m}U\right) + \mathrm{Tr}\left(E\,U^{-1}(USX_mS^{-1}U^{-1})_{-}U\right) \tag{3.43}$$

after use was made of cyclicity of the trace and $SES^{-1} = E$.

Let us now consider the second term on the right hand side of (3.43). Let $\mathcal{P}_{(-1)}$ be a projection on grade -1: $\mathcal{P}_{(-1)}(O) = O_{-1}$. It is clear that $\mathcal{P}_{(-1)}\left(U^{-1}\,(USX_mS^{-1}U^{-1})_{-}\,U\right) = \mathcal{P}_{(-1)}\left((USX_mS^{-1}U^{-1})_{-}\right)$. We can therefore rewrite (3.43) as:

$$\mathrm{Tr}\left(E\,S^{-1}\delta_{X_m}S\right) = -\,\mathrm{Tr}\left(E\,U^{-1}\delta_{X_m}U\right) + \mathrm{Tr}\left(E\,(USX_mS^{-1}U^{-1})_{-}\right) \tag{3.44}$$

The only non-zero contribution from the first term is $\mathrm{Tr}\left(E\delta_{X_m}u^{(-1)}\right)$ since $\delta_{X_m}u^{(-1)}$ is the only term in $U^{-1}\delta_{X_m}U$ of grade -1. However, since $u^{(-1)} \in \mathcal{M}$ the trace product of E with $\delta_{X_m}u^{(-1)}$ yields zero. Hence, there is no contribution from the first term on the right hand side of equation (3.44). Similarly, the remaining term on the right hand-side of eq. (3.44) is $\mathrm{Tr}\left(E\delta_{X_m}\mathfrak{s}^{(-1)}\right) = \delta_{X_m}\,\mathrm{Tr}\left(E\mathfrak{s}^{(-1)}\right)$. Hence, (3.44) is equivalent to:

$$\delta_{X_m}\,\mathrm{Tr}\,(E\,\mathfrak{s}) = \mathrm{Tr}\left(E\,(\Theta X_m\Theta^{-1})_{-}\right) \tag{3.45}$$

which is nothing but the statement that $\Omega(\cdot)$ is exact and reproduced by

$$\Omega(X_m) = -\delta_{X_m}\,\mathrm{Tr}\,(E\,\mathfrak{s}) = -\delta_{X_m}\,\mathrm{Tr}\left(E\,\mathfrak{s}^{(-1)}\right) \tag{3.46}$$

3.3 Isospectral Flows and Conservation Laws

For the special case of $X_n = b_n \in \mathcal{C}(\mathcal{K})$ definition 3.3 becomes :

$$\mathcal{J}_n \equiv \mathrm{Tr}\left([\,Q_{\mathbf{s}}\,,\,\Theta\,]\;b_n\Theta^{-1}\right) \tag{3.47}$$

$$\mathcal{H}_n \equiv \Omega(b_n) = -\,\mathrm{Tr}\left(EUb_nU^{-1}\right) \tag{3.48}$$

Note, that $\mathcal{J}_n$ depends explicitly on S and is therefore in general a non-local quantity in contrast to $\mathcal{H}_n$.

From Proposition 3.1 we obtain :

$$\partial_x\mathcal{J}_n = \mathcal{H}_n \tag{3.49}$$

Proposition 3.3

$$\frac{d}{dt_m}\mathcal{J}_n = -\,\mathrm{Tr}\left([\,Q_{\mathbf{s}}\,,\,B_m\,]\;\Theta\,b_n\,\Theta^{-1}\right) \tag{3.50}$$

Proof. The proof follows from the direct calculation:

$$\begin{aligned}\frac{d}{dt_m}\mathcal{J}_n &= \mathrm{Tr}\left([\,Q_{\mathbf{s}}\,,\,(\Theta b_m - B_m\Theta)\,]\;b_n\,\Theta^{-1}\right)\\ &+ \mathrm{Tr}\left([\,Q_{\mathbf{s}}\,,\,\Theta\,]\;b_n\,(-b_m\,\Theta^{-1} + \Theta^{-1}\,B_m\right)\end{aligned} \tag{3.51}$$

where we used relations (3.5)-(3.6). Eq. (3.50) follows now from commutativity of b_n and b_m and also due to the fact that $\mathrm{Tr}(nb_nb_m) = 0$. □

Note, that expression in equation (3.50) can be rewritten as

$$\frac{d}{dt_m}\mathcal{J}_n = -\,\mathrm{Tr}\left([\,Q_{\mathbf{s}}\,,\,B_m\,]\;U\,b_n\,U^{-1}\right) \tag{3.52}$$

which clearly exhibits the local character of $d\mathcal{J}_n/dt_m$.

From Proposition 3.2 applied to the abelian center of $\mathcal{K}$ follows a set of corollaries :

Corollary 3.1

$$\frac{d}{dt_m}\mathcal{J}_n = \frac{d}{dt_n}\mathcal{J}_m \tag{3.53}$$

$$\frac{d}{dt_m}\mathcal{H}_n = \frac{d}{dt_n}\mathcal{H}_m \tag{3.54}$$

Inserting expansions (2.11) and (2.12) into relation (3.50) we obtain

$$\begin{aligned}\frac{d}{dt_m}\mathcal{J}_n &= -m\,\mathrm{tr}\left(b_m\beta_n^{(-m)}\right) - \sum_{k=0}^{m-1}(m-k-1)\,\mathrm{tr}\left(\beta_m^{(m-k-1)}\beta_n^{(k+1-m)}\right)\\ \frac{d}{dt_n}\mathcal{J}_m &= -n\,\mathrm{tr}\left(b_n\beta_m^{(-n)}\right) - \sum_{k=0}^{n-1}(n-k-1)\,\mathrm{tr}\left(\beta_n^{(n-k-1)}\beta_m^{(k+1-n)}\right)\end{aligned}$$

Taking $m = 1$ in the above equations and equating them according to relation (3.53) we get the recurrence relation:

$$\mathrm{tr}\left(E\beta_n^{(-1)}\right) = n\,\mathrm{tr}\left(b_n\beta_1^{(-n)}\right) - \sum_{k=0}^{n-1}(n-k-1)\,\mathrm{tr}\left(\beta_n^{(n-k-1)}\beta_1^{(k+1-n)}\right) \tag{3.55}$$

A conservation law has the form

$$\frac{d}{dt_m}\mathcal{H}_n + \partial_x Q_{m,n} = 0 \tag{3.56}$$

with local, conserved flux $Q_{m,n}$ and conserved (Hamiltonian) density $\mathcal{H}_n$. Relations (3.49) and (3.52) establish existence of an infinite number of local conservation laws for all integrable models obtained by the algebraic dressing construction. The local, conserved flux is given by $Q_{m,n} = \mathrm{Tr}\left([\,Q_{\mathbf{s}}\,,\,B_m\,]\;U\,b_n\,U^{-1}\right)$. The following corollary follows easily.

Corollary 3.2 *The Hamiltonians defined by*

$$H_n = \int \mathcal{H}_n \, dx \quad ; \quad n = 1, 2, \ldots \tag{3.57}$$

are conserved.

Proof. Indeed

$$\frac{d}{dt_m} H_n = \int \partial_x \frac{d}{dt_m} \mathcal{J}_n \tag{3.58}$$

The desired result follows now recalling that the quantities $d\mathcal{J}_n/dt_m$ are local as shown in (3.52). □

Let us apply $\mathcal{H}_n = -\operatorname{Tr}(EUb_nU^{-1})$ to the special case of $n = 1$:

$$\begin{aligned} \mathcal{H}_1 &= -\operatorname{Tr}\left(EUEU^{-1}\right) = \operatorname{Tr}\left(E\left[E\,,\,u^{(-2)}\right]\right) \\ &- \frac{1}{2}\operatorname{Tr}\left(E\left[u^{(-1)}\,,\,\left[u^{(-1)}\,,\,E\right]\right]\right) \end{aligned} \tag{3.59}$$

By well-known trace identity the first term is zero and the second term becomes

$$\mathcal{H}_1 = \frac{1}{2}\operatorname{Tr}\left(\left[u^{(-1)}\,,\,E\right]^2\right) = \frac{1}{2}\operatorname{Tr}\left(A^2\right) \tag{3.60}$$

which is valid for models described by the Lax operator L from (2.7).

One important consequence of Corollary 3.1 is that $\mathcal{J}_n$ appears to be $\frac{d}{dt_n}$ of some function of phase variables. The standard way of writing this is in terms of the tau-function:

Definition 3.4

$$\mathcal{J}_n = -\frac{d}{dt_n} \log \tau \,. \tag{3.61}$$

We therefore have:

$$\operatorname{Tr}\left(U^{-1}\left[\,Q_{\mathfrak{s}}\,,\,U\,\right]\,b_n\right) + \operatorname{Tr}\left(S^{-1}\left[\,Q_{\mathfrak{s}}\,,\,S\,\right]\,b_n\right) = -\frac{d}{dt_n}\log\tau \tag{3.62}$$

Recall, that $U = \exp(\mathfrak{u})$, $S = \exp(\mathfrak{s})$ and

$$(de^{\mathfrak{s}})\,e^{-\mathfrak{s}} = \sum_{n=1}^{\infty} \frac{1}{n!}\,(\operatorname{ad}\mathfrak{s})^{n-1}\,d\mathfrak{s} \tag{3.63}$$

for a derivation d. Using this one can establish that the contribution on the left hand side of (3.62) from the second term is equal to $\operatorname{Tr}([\,Q_{\mathfrak{s}}\,,\,\mathfrak{s}\,]\,b_n)$. For $n = 1, 2$ the contribution on the left hand side of (3.62) from the first term is zero and accordingly

$$\operatorname{tr}\left(\mathfrak{s}^{(-1)}E\right) = \partial_x \log\tau \tag{3.64}$$

$$\operatorname{tr}\left(\mathfrak{s}^{(-2)}b_2\right) = \frac{1}{2}\frac{d}{dt_2}\log\tau \tag{3.65}$$

One important consequence of (3.31) is that

$$\mathcal{H}_n = -\frac{d}{dt_n}\partial_x \log\tau = -\frac{d}{dt_n}\operatorname{tr}\left(\mathfrak{s}^{(-1)}E\right) \tag{3.66}$$

where in the last equality we used eq.(3.64). This is in perfect agreement with eq.(3.46) obtained in different way in a general case.

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3.3.1 Homogeneous Gradation

We now turn to the homogeneous gradation with $E = \lambda E^{(0)}$, $b_n \equiv E^{(n)} = \lambda^n E^{(0)}$. In this setup we are working with expansions :

$$\Theta = 1 + \sum_{k=1}^{\infty} \theta^{(-k)}/\lambda^k \tag{3.67}$$

$$\Theta E \Theta^{-1} = E + A + \sum_{k=1}^{\infty} A^{(-k)}/\lambda^k \tag{3.68}$$

$$\Theta b_n \Theta^{-1} = \lambda^{n-1}E + \lambda^{n-1}A + \sum_{k=1}^{\infty} \lambda^{n-k-1}A^{(-k)} \tag{3.69}$$

It follows that:

$$B_n = b_n + \lambda^{n-1}A + \sum_{k=1}^{n-1} \lambda^{n-k-1}A^{(-k)} \tag{3.70}$$

Comparing with expansion (2.11) we see that $\beta_n^{(j)} = \lambda^j A^{j+1-n}$.

In case of homogeneous gradation the definition (3.47) becomes :

$$\mathcal{J}_n = \operatorname{Tr}\left(\frac{d\Theta}{d\lambda} b_{n+1}\Theta^{-1}\right) \tag{3.71}$$

while (3.50) simplifies to :

$$\frac{d}{dt_m}\mathcal{J}_n = -\operatorname{Tr}\left(\frac{dB_m}{d\lambda}\,\Theta\,b_{n+1}\,\Theta^{-1}\right) \tag{3.72}$$

Expanding the right hand side of (3.72) we obtain (with $A^{(0)} = A$) :

$$\frac{d}{dt_m}\mathcal{J}_n = -m\operatorname{tr}\left(E^{(0)}A^{(1-n-m)}\right) - \sum_{k=0}^{m-1}(m-k-1)\operatorname{tr}\left(A^{(-k)}A^{(2+k-n-m)}\right) \tag{3.73}$$

which is equal to :

$$\frac{d}{dt_n}\mathcal{J}_m = -n\operatorname{tr}\left(E^{(0)}A^{(1-n-m)}\right) - \sum_{k=0}^{n-1}(n-k-1)\operatorname{tr}\left(A^{(-k)}A^{(2+k-n-m)}\right)$$

Another crucial observation is that :

$$A^{(-n)} = \frac{d\,\theta^{(-1)}}{dt_n} \tag{3.74}$$

The proof follows by projecting eq.(3.5) on the -1 grade. This gives according to (3.69) :

$$\frac{d}{dt_n}\theta^{(-1)}\lambda^{-1} = \left(\Theta b_n \Theta^{-1}\right)_{-1} = A^{(-n)}\lambda^{-1} \tag{3.75}$$

which leads to the desired relation.

One consequence of (3.74) is a relation

$$B_n = b_n + \lambda^{n-1} A + \sum_{k=1}^{n-1} \lambda^{n-k-1} \frac{d\,\theta^{(-1)}}{dt_k} \tag{3.76}$$

from which follows the recurrence relation:

$$B_{n+1} = \lambda B_n + \frac{d\,\theta^{(-1)}}{dt_n} = \lambda B_n + A^{(-n)} \tag{3.77}$$

Also

$$\partial_x \mathcal{J}_n = \mathcal{H}_n = -\,\mathrm{tr}\left(E^{(0)} A^{(-n)}\right) = -\,\mathrm{tr}\left(E^{(0)} \frac{d\theta^{(-1)}}{dt_n}\right) \tag{3.78}$$

Note, that the Hamiltonian densities in the the homogeneous gradation can be rewritten as :

$$\mathcal{H}_n = -\,\mathrm{Tr}\left(EUE^{(n)}U^{-1}\right) = -\,\mathrm{Tr}\left(E^{(0)}UE^{(n+1)}U^{-1}\right) = -\,\mathrm{Tr}\left(E^{(0)}B_{n+1}\right) \tag{3.79}$$

c.f. [4, 12, 13]. Consider the matrix A as in (6.18). Then the general expression for $\mathcal{H}_1$ found in (3.60) becomes in this case $\mathcal{H}_1 = \sum_i^M q_i r_i$. For the case of $n = 2$ ($\mathcal{H}_2$) we obtain :

$$\mathcal{H}_2 = \mathrm{Tr}\left([u^{(-2)}, E^{(2)}][u^{(-1)}, E]\right) = \sum_{i=1}^{M} (q_i r_{i\,,x} - q_{i\,,x} r_i)$$

which is consistent with $\partial_x \mathcal{H}_2 = d\mathcal{H}_1/dt_2$.

Plugging $\beta_n^{(j)} = \lambda^j A^{(j+1-n)}$ into the recurrence relation (3.55) we obtain

$$\mathrm{tr}\left(E^{(0)}A^{(-n)}\right) = n\,\mathrm{tr}\left(E^{(0)}A^{(-n)}\right) + \sum_{k=0}^{n-1}(n-k-1)\,\mathrm{tr}\left(A^{(-k)}A^{(1+k-n)}\right)$$

This recurrence relation can equivalently be written as :

$$\mathrm{tr}\left(E^{(0)}A^{(-n)}\right) = -\,\mathrm{tr}\left(AA^{(1-n)}\right) - \frac{1}{2}\sum_{k=1}^{n-2}\mathrm{tr}\left(A^{(-k)}A^{(1+k-n)}\right) \quad \text{for } n > 1 \tag{3.80}$$

3.4 Recursion Relations

From (3.5) we find :

$$\delta_{b_N}\left(\Theta b_M \Theta^{-1}\right) = -\left[\, B_N\,,\, \left(\Theta b_M \Theta^{-1}\right)\right] \tag{3.81}$$

Of special interest is $b_1 = E$ and the corresponding conjugated element $(\Theta b_N \Theta^{-1})$, which is given by expansion in grading (see eq.(2.11)):

$$\begin{aligned} \Theta b_1 \Theta^{-1} = UEU^{-1} &= E + \left[\, u^{(-1)}\,,\, E\right] + \sum_{k=1}^{\infty}\beta^{(-k)} \\ &= E + A + \sum_{k=1}^{\infty}\beta^{(-k)} \end{aligned} \tag{3.82}$$

where in the last equation we used that by construction $\left[\, u^{(-1)}\,,\, E\right] = A$ and for brevity we wrote $\beta_1^{(-k)} = \beta^{(-k)}$. Plugging (3.82) into (3.81) we obtain by projecting on grade zero :

$$\delta_{b_N} A = -\left[\, b_N\,,\, \beta^{(-N)}\right] \tag{3.83}$$

Hence, only $\mathcal{M}$ components of $\beta^{(-N)}$ will make a non-zero contribution to the flows of A. For $N = 1$ we find from eq. (3.83):

$$\delta_1 A = \partial_x A = -\left[\, E\,,\, \beta^{(-1)}\right] \tag{3.84}$$

while from eq. (3.81) we get

$$\delta_1 \beta^{(-n)} = \partial_x \beta^{(-n)} = -\left[\, E\,,\, \beta^{-(n+1)}\right] - \left[\, A\,,\, \beta^{(-n)}\right] \tag{3.85}$$

Introducing, the covariant derivative $\mathcal{D} = \partial_x + ad_A$ we can rewrite (3.85) in a compact form as:

$$\mathcal{D}\beta^{(-n)} + ad_E\left(\beta^{-(n+1)}\right) = 0 \tag{3.86}$$

which decomposes on $\mathcal{M}$ and $\mathcal{K}$ directions (with $\beta^{(-n)} = \beta_{\mathcal{M}}^{(-n)} + \beta_{\mathcal{K}}^{(-n)}$) as follows:

$$\beta_{\mathcal{M}}^{-(n+1)} = -ad_E^{-1}\left((\mathcal{D}\beta^{(-n)})|_{\mathcal{M}}\right) \quad ; \quad (\mathcal{D}\beta^{(-n)})|_{\mathcal{K}} = 0 \tag{3.87}$$

The first of expressions in (3.87) can be rewritten as:

$$\begin{aligned} \beta_{\mathcal{M}}^{(-n-1)} &= -ad_E^{-1}\left(\partial_x \beta_{\mathcal{M}}^{(-n)} + [A\,,\, \beta_{\mathcal{K}}^{(-n)}] + [A\,,\, \beta_{\mathcal{M}}^{(-n)}]|_{\mathcal{M}}\right) \\ &= -ad_E^{-1}\left(\partial_x \beta_{\mathcal{M}}^{(-n)} - [A\,,\, \partial_x^{-1}\left([A\,,\, \beta_{\mathcal{M}}^{(-n)}]|_{\mathcal{K}}\right)] + [A\,,\, \beta_{\mathcal{M}}^{(-n)}]|_{\mathcal{M}}\right) \end{aligned} \tag{3.88}$$

where we substituted $\beta_{\mathcal{K}}^{(-n)}$ by:

$$\beta_{\mathcal{K}}^{(-n)} = -\partial_x^{-1}\left([A\,,\, \beta_{\mathcal{M}}^{(-n)}]|_{\mathcal{K}}\right) \tag{3.89}$$

derived from the second equation in (3.87).

Since $[A\,,\, \beta_{\mathcal{M}}^{(-n)}]|_{\mathcal{K}} = \left(\mathcal{D}\beta_{\mathcal{M}}^{(-n)}\right)|_{\mathcal{K}}$ we can rewrite (3.88) as

$$\beta_{\mathcal{M}}^{(-n-1)} = \mathcal{R}\left(\beta_{\mathcal{M}}^{(-n)}\right) \tag{3.90}$$

with help of the recursion operator:

$$\mathcal{R} = -ad_E^{-1}\left(\Pi_{\mathcal{M}}\mathcal{D} - ad_A \partial_x^{-1}\Pi_{\mathcal{K}}\mathcal{D}\right) \tag{3.91}$$

where $\Pi_{\mathcal{M}}, \Pi_{\mathcal{K}}$ are projections on $\mathcal{M}$ and $\mathcal{K}$ spaces. This construction simplifies significantly when considered in case of the homogeneous gradation (and symmetric spaces with $[\,\mathcal{M}\,,\,\mathcal{M}\,] \subset \mathcal{K}$). From now on we consider therefore the special case of homogeneous gradation In this case we have expansion in (3.70). Recall, that for symmetric spaces $ad_E^2 = \lambda^2 I$ on $\mathcal{M}$. By applying ad_E on both sides of eq. (3.85) we obtain:

$$A^{(-n-1)}|_{\mathcal{M}} = -\left[\, E^{(0)}\,,\, \partial_x A^{(-n)} + \left[\, A\,,\, A^{(-n)}\right]\right] = -ad_{E^{(0)}}(\mathcal{D}A^{(-n)}) \tag{3.92}$$

where $E^{(0)} = \lambda^{-1}E$. For the $\mathcal{K}$ component we get from (3.85) a non-local expression:

$$A^{(-n-1)}|_{\mathcal{K}} = -\partial_x^{-1}\left[A\,,\,A^{(-n)}|_{\mathcal{M}}\right] \tag{3.93}$$

From above equations obtain the recurrence relation:

$$A^{(-n-1)}|_{\mathcal{M}} = \mathcal{R}(A^{(-n)}|_{\mathcal{M}}) \tag{3.94}$$

with help of the recursion operator:

$$\mathcal{R} = -ad_E^{-1}\left(\partial - ad_A\partial_x^{-1}ad_A\right) \tag{3.95}$$

which is a specialization of $\mathcal{R}$ in eq. (3.91) in case of symmetric spaces. Also, from (3.84) we get :

$$A^{(-1)}|_{\mathcal{M}} = -ad_{E^{(0)}}(\partial_x A) = \mathcal{R}(A)) \tag{3.96}$$

and therefore:

$$A^{(-n)}|_{\mathcal{M}} = \mathcal{R}^n(A)) \tag{3.97}$$

which is a well-known recursion relation.

3.5 Example: AKNS Hierarchy; The Homogeneous Hierarchy with $\hat{\mathbf{sl}}(\mathbf{2}) = \mathbf{A}_\mathbf{1}^{(\mathbf{1})}$

We take $\mathcal{G} = sl(2,\mathbb{C})$ with standard basis $e = \sigma_+$, $f = \sigma_-$ and $h = \sigma_3$. The operator $L = D + E + A$ reads:

$$L = \begin{pmatrix} D+\lambda/2 & q \\ r & D-\lambda/2 \end{pmatrix} = I\cdot D + \frac{\lambda}{2}h + qe + rf \tag{3.98}$$

The matrix $U = \exp\left(\sum_{j\geq 1} u^{(-j)}\lambda^{-j}\right)$ with

$$u^{(-1)} = \begin{pmatrix} 0 & -q \\ r & 0 \end{pmatrix}\;;\; u^{(-2)} = \begin{pmatrix} 0 & q_x \\ r_x & 0 \end{pmatrix}\;;\;\ldots \tag{3.99}$$

transforms L as follows :

$$U^{-1}LU = \begin{pmatrix} D+\lambda/2 & 0 \\ 0 & D-\lambda/2 \end{pmatrix} + \sum_{i=1}^{\infty} k^{(-i)}\lambda^{-i}\sigma_3 \tag{3.100}$$

where to lowest orders in λ^{-1} we find:

$$\begin{aligned} \sum_{i=1}^{\infty} k^{(-i)}\lambda^{-i}\sigma_3 &= \begin{pmatrix} qr & 0 \\ 0 & -qr \end{pmatrix}\lambda^{-1} \\ &+ \frac{1}{2}\begin{pmatrix} -q_x r + q r_x & 0 \\ 0 & -q r_x + r q_x \end{pmatrix}\lambda^{-2} + O(\lambda^{-3}) \end{aligned} \tag{3.101}$$

We obtain the following expression for B_2:

$$B_2 = (Ub_2U^{-1})_+ = \begin{pmatrix} \lambda^2/2 - qr & \lambda q - q_x \\ \lambda r + r_x & -\lambda^2/2 + qr \end{pmatrix} \tag{3.102}$$

The corresponding flows :

$$\partial_2 q = -q_{xx} + 2q^2 r \quad ; \quad \partial_2 r = r_{xx} - 2qr^2 \tag{3.103}$$

reproduce the well-known Nonlinear Schrödinger (NLS) equation.

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3.5.1 Tau Functions from the Squared Eigenfunction Potentials

Let standard AKNS pseudo-differential Lax operator be

$$\mathcal{L} = D + \Phi D^{-1}\Psi \tag{3.104}$$

The linear problem $\mathcal{L}\psi_{BA} = \lambda\psi_{BA}$ can be decomposed as

$$\partial_x\psi_{BA} + \Phi S(t,\lambda) = \lambda\psi_{BA} \;\; ; \;\; \partial_x S(t,\lambda) = \Psi\psi_{BA}(t,\lambda) \tag{3.105}$$

Similarly, we can introduce the conjugated linear problem: $\mathcal{L}^*\psi^*_{BA} = (-D - \Psi D^{-1}\Phi)\psi^*_{BA} = \lambda\psi^*_{BA}$ which can be rewritten as

$$\partial_x S^*(t,\lambda) = \Phi\psi^*_{BA} \;\; ; \;\; -\partial_x\psi^*_{BA}(t,\lambda) - \Psi S^*(t,\lambda) = \lambda\psi^*_{BA} \tag{3.106}$$

Recall from [14], that in the Sato formalism the squared eigenfunction potentials $S(t,\lambda), S^*(t,\lambda)$ are given by:

$$S(t,\lambda) = \frac{1}{\lambda}\Psi(t-[\lambda^{-1}])\frac{\tau(t-[\lambda^{-1}])}{\tau(t)}e^{\xi(t,\lambda)} \tag{3.107}$$

$$S^*(t,\lambda) = -\frac{1}{\lambda}\Phi(t+[\lambda^{-1}])\frac{\tau(t+[\lambda^{-1}])}{\tau(t)}e^{-\xi(t,\lambda)} \tag{3.108}$$

where $\xi(t,\lambda) = \sum \lambda^j t_j$.

We compare the above pseudo-differential setup to the algebraic dressing formalism. Consider, the matrix $L_0 \equiv D + E = D + \lambda\sigma_3/2$ obtained by "un-dressing" the matrix Lax operator L from eq.(3.98). Let

$$\Psi^+_{vac} = e^{-\sum_i E^{(i)}t_i}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}e^{-\xi(t,\lambda)/2} \tag{3.109}$$

$$\Psi^-_{vac} = e^{-\sum_i E^{(i)}t_i}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}e^{\xi(t,\lambda)/2} \tag{3.110}$$

be two solutions of equation $L_0\Psi_0 = 0$. Since, $L_0 = S^{-1}U^{-1}LUS$ it follows that $\Psi^\pm = US\Psi^\pm_{vac} = \Theta\Psi^\pm_{vac}$ satisfy $L\Psi^\pm = 0$.

Let us write Θ as a 2×2 matrix:

$$\Theta = \begin{pmatrix}\theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22}\end{pmatrix} \tag{3.111}$$

then

$$\begin{aligned} L\Psi^- &= \begin{pmatrix} D+\lambda/2 & q(t) \\ r(t) & D-\lambda/2 \end{pmatrix}\begin{pmatrix}\theta_{12}\\ \theta_{22}\end{pmatrix}e^{\xi(t,\lambda)/2} = 0 \\ &\to \begin{pmatrix} D & q(t) \\ r(t) & D-\lambda \end{pmatrix}\begin{pmatrix}\theta_{12}\\ \theta_{22}\end{pmatrix}e^{\xi(t,\lambda)} = 0 \end{aligned} \tag{3.112}$$

The last equation can be cast into the form of

$$\theta_{12}e^{\xi(t,\lambda)} = -\partial^{-1}\left(q\theta_{22}e^{\xi(t,\lambda)}\right) \tag{3.113}$$

$$\lambda\theta_{22}e^{\xi(t,\lambda)} = \left(\partial - r\partial^{-1}q\right)\theta_{22}e^{\xi(t,\lambda)} \tag{3.114}$$

Comparing with (3.105) while making an identification

$$r = \Phi \qquad ; \qquad q = -\Psi \tag{3.115}$$

we find

$$\theta_{22}e^{\xi(t,\lambda)} = \psi_{BA}(t,\lambda) = \frac{\tau(t-[\lambda^{-1}])}{\tau(t)}e^{\xi(t,\lambda)}$$

$$\theta_{12}e^{\xi(t,\lambda)} = -S\left(q,\psi_{BA}(t,\lambda)\right) = -\frac{q(t-[\lambda^{-1}])\tau(t-[\lambda^{-1}])}{\lambda\tau(t)}e^{\xi(t,\lambda)}$$

Similarly,

$$L\Psi^{+} = \begin{pmatrix} D+\lambda/2 & q(t) \\ r(t) & D-\lambda/2 \end{pmatrix}\begin{pmatrix} \theta_{11} \\ \theta_{21} \end{pmatrix} e^{-\xi(t,\lambda)/2} = 0$$

$$\rightarrow \begin{pmatrix} D+\lambda & q(t) \\ r(t) & D \end{pmatrix}\begin{pmatrix} \theta_{11} \\ \theta_{21} \end{pmatrix} e^{-\xi(t,\lambda)} = 0 \tag{3.116}$$

The last equation can be cast into the form of

$$\theta_{21}e^{-\xi(t,\lambda)} = -\partial^{-1}\left(r\theta_{11}e^{-\xi(t,\lambda)}\right) \tag{3.117}$$

$$\lambda\theta_{11}e^{-\xi(t,\lambda)} = \left(\partial - r\partial^{-1}q\right)^{*}\theta_{11}e^{-\xi(t,\lambda)} \tag{3.118}$$

Comparing with (3.106) and (3.115) we find:

$$\theta_{11}e^{-\xi(t,\lambda)} = \psi^{*}_{BA}(t,\lambda) = \frac{\tau(t+[\lambda^{-1}])}{\tau(t)}e^{-\xi(t,\lambda)}$$

$$\theta_{21}e^{-\xi(t,\lambda)} = -S\left(r,\psi^{*}_{BA}(t,\lambda)\right) = \frac{r(t+[\lambda^{-1}])\tau(t+[\lambda^{-1}])}{\lambda\tau(t)}e^{-\xi(t,\lambda)}$$

In this way we obtain the explicit matrix form of the matrices Θ and Θ^{-1} in terms of the τ function:

$$\Theta = \frac{1}{\tau(t)}\begin{pmatrix} \tau(t_{+}(\lambda)) & -\frac{1}{\lambda}q(t_{-}(\lambda)))\tau(t_{-}(\lambda)) \\ \frac{1}{\lambda}r(t_{+}(\lambda))\tau(t_{+}(\lambda)) & \tau(t_{-}(\lambda)) \end{pmatrix} \tag{3.119}$$

$$\Theta^{-1} = \frac{1}{\tau(t)}\begin{pmatrix} \tau(t_{+}(\lambda)) & \frac{1}{\lambda}q(t_{-}(\lambda))\tau(t_{-}(\lambda)) \\ -\frac{1}{\lambda}r(t_{+}(\lambda))\tau(t_{+}(\lambda)) & \tau(t_{-}(\lambda)) \end{pmatrix} \tag{3.120}$$

with $t_{\pm}(\lambda) \equiv t \pm [\lambda^{-1}] = (t_1 \pm 1/\lambda, t_2 \pm 1/2\lambda^2, \ldots)$. These expressions agree with the result of [15] obtained within Wilson's framework [16, 4]. The condition $\det\Theta = 1$ implies:

$$1 = \frac{\tau(t+[\lambda^{-1}])\tau(t-[\lambda^{-1}])}{\tau^2(t)}\left(1 + \frac{q(t-[\lambda^{-1}])r(t+[\lambda^{-1}])}{\lambda^2}\right) \tag{3.121}$$

or, equivalently $\psi_{BA}(t,\lambda)\psi^{*}_{BA}(t,\lambda) + S(t,\lambda)S^{*}(t,\lambda) = 1$.

Writing U as $U = \exp(u_{+}(t,\lambda)\sigma_{+} + u_{-}(t,\lambda)\sigma_{-})\exp(\mathfrak{s}(t,\lambda)\sigma_3)$ and comparing with eq.(3.119) we obtain :

$$\cosh^2\left(\sqrt{u_{+}u_{-}}\right) = \frac{\tau(t-[\lambda^{-1}])\,\tau(t+[\lambda^{-1}])}{\tau^2(t)} \tag{3.122}$$

and

$$e^{2\mathfrak{s}(\lambda)} = \frac{\tau(t+[\lambda^{-1}])}{\tau(t-[\lambda^{-1}])} \quad\rightarrow\quad \mathfrak{s} = \sum_{i=1}^{\infty}\mathfrak{s}^{(-i)} = \frac{1}{2}\ln\frac{\tau(t+[\lambda^{-1}])}{\tau(t-[\lambda^{-1}])} \tag{3.123}$$

or in terms of Schur polynomials :

$$\mathfrak{s}^{(-n)} = -\frac{1}{2\lambda^n}\left(p_n(-[\partial]) - p_n([\partial])\right)\ln\tau(t) \;\; ; \;\; n \geq 1 \tag{3.124}$$

3.5.2 Hamiltonian Densities and the Tau Function of the AKNS Model

In case of AKNS model quantities $\mathcal{H}_n, \mathcal{J}_n$ become

$$\mathcal{H}_n = -\operatorname{Tr}\left(\lambda^{n+1}\frac{\sigma_3}{2}\Theta\frac{\sigma_3}{2}\Theta^{-1}\right) \quad , \quad \mathcal{J}_n = -\operatorname{Tr}\left(\lambda^{n+1}\Theta_{\lambda}\frac{\sigma_3}{2}\Theta^{-1}\right) \tag{3.125}$$

where we introduced the notation $f_\lambda = df = \lambda df/d\lambda$.

Expressions (3.119) and (3.120) allow to calculate

$$\Theta\frac{\sigma_3}{2}\Theta^{-1} = -\frac{1}{2}\sigma_3 \tag{3.126}$$

$$+ \;\; \frac{1}{\tau^2(t)}\begin{pmatrix} \tau(t_{+}(\lambda))\tau(t_{-}(\lambda)) & \frac{1}{\lambda}q(t_{-}(\lambda))\tau(t_{-}(\lambda))\tau(t_{+}(\lambda)) \\ \frac{1}{\lambda}r(t_{+}(\lambda))\tau(t_{-}(\lambda))\tau(t_{+}(\lambda)) & \tau(t_{+}(\lambda))\tau(t_{-}(\lambda)) \end{pmatrix}$$

which results in

$$\operatorname{Tr}\left(\frac{\sigma_3}{2}\Theta\frac{\sigma_3}{2}\Theta^{-1}\right) = \frac{\tau(t+[\lambda^{-1}])\tau(t-[\lambda^{-1}])}{\tau^2(t)} - \frac{1}{2} \tag{3.127}$$

On the other hand from definition (3.125) we have :

$$\operatorname{Tr}\left(\frac{\sigma_3}{2}\Theta\frac{\sigma_3}{2}\Theta^{-1}\right) = -\sum_{n=1}^{\infty}\mathcal{H}_n\lambda^{-n-1} + \frac{1}{2} \tag{3.128}$$

Comparing the last two equations we find that the following must hold

$$\frac{\tau(t+[\lambda^{-1}])\tau(t-[\lambda^{-1}])}{\tau^2(t)} = \sum_{n=1}^{\infty}\frac{d}{dt_n}\partial_x\log(\tau)\lambda^{-n-1} + 1 \tag{3.129}$$

This is equivalent to the Hirota equations:

$$\left(\frac{1}{2}D_1 D_n - p_{n+1}([D])\right)\tau\cdot\tau = 0 \tag{3.130}$$

due to identities:

$$\begin{aligned}
\frac{\tau(t_+(\lambda))\tau(t_-(\lambda))}{\tau^2(t)} &= \frac{1}{\tau^2(t_i)}\exp\left(\sum_{k=1}^{\infty}\frac{\partial}{k\lambda^k\partial\epsilon_k}\right)\tau(t+\epsilon)\,\tau(t-\epsilon)\,|_{\epsilon=0} \\
&= \frac{1}{\tau^2(t)}\sum_{k=0}^{\infty}\frac{p_k([D])\,\tau\cdot\tau}{\lambda^k} \qquad (3.131)\\
\frac{1}{2\tau^2(t)}D_1D_{n-1}\tau\cdot\tau &= \partial_x\partial_{n-1}\ln\tau \qquad (3.132)
\end{aligned}$$

where we used Hirota's operators defined by

$$D_j^m a\cdot b = \frac{\partial^m}{\partial s_j^m}a(t_j+s_j)b(t_j-s_j)|_{s_j=0} \qquad (3.133)$$

One can show that:

$$\begin{aligned}
\mathrm{Tr}\left(\Theta_\lambda\frac{\sigma_3}{2}\Theta^{-1}\right) &= \frac{1}{2\tau^2(t)}\left(\tau_\lambda(t_+(\lambda))\tau(t_-(\lambda)) - \tau(t_+(\lambda))\tau_\lambda(t_-(\lambda))\right. \\
&- \frac{1}{\lambda^2}(q(t_-(\lambda))\tau(t_-(\lambda)))_\lambda r(t_+(\lambda))\tau(t_+(\lambda)) \\
&+ \left.\frac{1}{\lambda^2}q(t_-(\lambda))\tau(t_-(\lambda))(r(t_+(\lambda))\tau(t_+(\lambda)))_\lambda\right) \qquad (3.134)
\end{aligned}$$

Observe, now that

$$f_\lambda(t\pm[\lambda^{-1}]) = \mp\sum_{k=0}^{\infty}\lambda^{-k}\frac{d}{dt_k}f(t\pm[\lambda^{-1}]) \qquad (3.135)$$

and therefore $\mathrm{Tr}\left(\Theta_\lambda\frac{\sigma_3}{2}\Theta^{-1}\right)$ can be rewritten as:

$$\begin{aligned}
&\frac{-1}{2\tau^2(t)}\sum_{k=0}^{\infty}\lambda^{-k}\frac{d}{dt_k}\tau(t_+(\lambda))\tau(t_-(\lambda))\left(1+\frac{q(t_-(\lambda))r(t_+(\lambda))}{\lambda^2}\right)\\
= &\frac{-1}{2\tau^2(t)}\sum_{k=0}^{\infty}\lambda^{-k}\frac{d}{dt_k}\tau^2(t) \qquad (3.136)
\end{aligned}$$

where use was made of condition (3.121). We therefore find

$$\mathrm{Tr}\left(\Theta_\lambda\frac{\sigma_3}{2}\Theta^{-1}\right) = -\sum_{k=0}^{\infty}\lambda^{-k}\frac{d}{dt_k}\log\tau(t) \qquad (3.137)$$

in agreement with eq. (3.62).

Recall, that $\partial_x\mathfrak{s}^{(-1)} = -k^{(-1)} = -qr$ and $\theta^{(-1)} = u^{(-1)} + \mathfrak{s}^{(-1)}\sigma_3$. Accordingly,

$$A^{(-1)} = \frac{d}{dt_1}\theta^{(-1)} = \partial_x\theta^{(-1)} = \partial_x\begin{pmatrix}0 & -q\\ r & 0\end{pmatrix} - qr\sigma_3 \qquad (3.138)$$

which is equal to :

$$\left(\Theta E\Theta^{-1}\right)_{-1} = \left[u^{(-2)}\,,\,E^{(0)}\right] + \frac{1}{2}\left[u^{(-1)}\,,\,\left[u^{(-1)}\,,\,E^{(0)}\right]\right] \qquad (3.139)$$

in agreement with (3.75).

Inserting $E^{(0)} = \sigma_3/2$ and

$$A^{(-k)} = \frac{d}{dt_k}\theta^{(-1)} = \frac{d}{d\,t_k}\left(\begin{pmatrix}0 & -q\\ r & 0\end{pmatrix} + \mathfrak{s}^{(-1)}\sigma_3\right) \qquad (3.140)$$

into the relation (3.80) we obtain (for $n>1$):

$$\frac{d\,\mathfrak{s}^{(-1)}}{d\,t_n} = r\frac{d\,q}{d\,t_{n-1}} - q\frac{d\,r}{d\,t_{n-1}} + \sum_{k=1}^{n-2}\left(\frac{d\,q}{d\,t_k}\frac{d\,r}{d\,t_{n-k-1}} - \frac{d\,\mathfrak{s}^{(-1)}}{d\,t_k}\frac{d\,\mathfrak{s}^{(-1)}}{d\,t_{n-k-1}}\right)$$

in agreement with reference [17].

Recall, that $\mathcal{H}_n = -d\mathfrak{s}^{(-1)}/dt_n$. Accordingly, the above equation becomes a recurrence relation for the Hamiltonian densities of the AKNS model :

$$\mathcal{H}_n = -r\frac{d\,q}{d\,t_{n-1}} + q\frac{d\,r}{d\,t_{n-1}} - \sum_{k=1}^{n-2}\frac{d\,q}{d\,t_k}\frac{d\,r}{d\,t_{n-k-1}} + 2rq\frac{d\,\mathfrak{s}^{(-1)}}{d\,t_{n-2}} + \sum_{k=2}^{n-3}\mathcal{H}_k\mathcal{H}_{n-k-1}$$

3.6 Non-abelian Symmetries of the Integrable Models, sl(3) Example

One of advantages of the dressing approach is that it provides a convenient framework to classify and describe the symmetries of integrable models. In particular, the non-abelian symmetries emerge naturally in this framework for models with the non-abelian kernel $\mathcal{K}$ of ad E. To illustrate the non-abelian symmetry structure of such models we consider here the linear spectral problem based on $sl(3)$ Lie algebra with the homogeneous gradation $Q_{\mathfrak{s}} \equiv d$. Here, the semi-simple and non-regular grade-one element E is given by:

$$E = H^{(1)}_{\mu_2} = \frac{\lambda}{3}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix} \qquad (3.141)$$

Thus the kernel $\mathcal{K}$ is the non-abelian sub-algebra $\{\widehat{sl}(2)\oplus\hat{U}(1)\}$ of $\widehat{\mathcal{G}} = \widehat{sl}(3)$ spanned by:

$$\mathcal{K} = \left\{E^{(n)} \equiv \lambda^n H_{\mu_2},\,\lambda^n H_{\mu_1},\,\lambda^n E_{\pm\alpha_1}\right\} \qquad (3.142)$$

where

$$H_{\mu_1} = \frac{1}{3}\begin{pmatrix}2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix} \qquad (3.143)$$

and in the Weyl basis $E_{\alpha_1\,ij} = \delta_{i1}\delta_{j2}$ and $E_{-\alpha_1\,ij} = \delta_{i2}\delta_{j1}$. The center $\mathcal{C}(\mathcal{K}) = \{E^{(n)}\} = \hat{U}(1)$ is spanned by one element H_{μ_2} only. The image is given by $\mathcal{M} = \left\{E^{(n)}_{\pm\alpha_2},\,E^{(n)}_{\pm(\alpha_1+\alpha_2)}\right\}$. Accordingly, the Lax operator is:

$$L = D\cdot I + E + A = D\cdot I + \frac{\lambda}{3}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix} + \begin{pmatrix}0 & 0 & q_1\\ 0 & 0 & q_2\\ r_1 & r_2 & 0\end{pmatrix} \qquad (3.144)$$

with the matrix $A \in \mathcal{M}_0$.

The dressing procedure

$$U^{-1} L U = D \cdot I + \frac{\lambda}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + k_1 \lambda^{-1} + O(\lambda^{-2}) \tag{3.145}$$

holds to the lowest order with

$$k_1 = \begin{pmatrix} q_1 r_1 & q_1 r_2 & 0 \\ q_2 r_1 & q_2 r_2 & 0 \\ 0 & 0 & -q_1 r_1 - q_2 r_2 \end{pmatrix} \tag{3.146}$$

and

$$u^{(-1)} = \begin{pmatrix} 0 & 0 & -q_1 \\ 0 & 0 & -q_2 \\ r_1 & r_2 & 0 \end{pmatrix} \tag{3.147}$$

in $U = \exp(u^{(-1)} \lambda^{-1} + O(\lambda^{-2}))$. We now apply these results to calculate the symmetry transformations :

$$\delta^{(1)}_{\pm\alpha_1} A \equiv [L \, , \, (\Theta \lambda^1 E_{\pm\alpha_1} \Theta^{-1})_+] = [\partial_x + A \, , \, (\Theta \lambda^1 E_{\pm\alpha_1} \Theta^{-1})_0] \tag{3.148}$$

where

$$(\Theta \lambda^1 E_{\pm\alpha_1} \Theta^{-1})_0 = [s^{(-1)} \, , \, E_{\pm\alpha_1}] + [u^{(-1)} \, , \, E_{\pm\alpha_1}] \tag{3.149}$$

and $s^{(-1)} = -\partial^{-1}(k_1)$. The transformations (3.148) are in components given by :

$$\delta^{(1)}_{\alpha_1}(q_1) = q_2' - q_1 \partial^{-1}(q_2 r_1) - q_2 \partial^{-1}(q_2 r_2 - q_1 r_1) \; ; \; \delta^{(1)}_{\alpha_1}(q_2) = q_2 \partial^{-1}(q_2 r_1)$$
$$\delta^{(1)}_{\alpha_1}(r_1) = r_1 \partial^{-1}(q_2 r_1) \; ; \; \delta^{(1)}_{\alpha_1}(r_2) = r_1' - r_2 \partial^{-1}(q_2 r_1) + r_1 \partial^{-1}(q_2 r_2 - q_1 r_1)$$

and

$$\delta^{(1)}_{-\alpha_1}(q_1) = q_1 \partial^{-1}(q_1 r_2) \; ; \; \delta^{(1)}_{-\alpha_1}(q_2) = q_1' - q_2 \partial^{-1}(q_1 r_2) + q_1 \partial^{-1}(q_2 r_2 - q_1 r_1)$$
$$\delta^{(1)}_{-\alpha_1}(r_1) = r_2' - r_1 \partial^{-1}(q_1 r_2) - r_2 \partial^{-1}(q_2 r_2 - q_1 r_1) \; ; \; \delta^{(1)}_{-\alpha_1}(r_2) = r_2 \partial^{-1}(q_1 r_2)$$

These results can be reproduced compactly by a much simpler formula in the framework based on the pseudo-differential Lax operator. To demonstrate this we note that the matrix spectral problem $L\Psi = 0$ with L from eq.(3.144) can be reformulated in an equivalent form as the scalar spectral problem :

$$\mathcal{L} \psi_{BA} = \lambda \psi_{BA} \quad ; \quad \mathcal{L} = D - r_1 D^{-1} q_1 - r_2 D^{-1} q_2 \tag{3.150}$$

Define :

$$\mathcal{M}_X = \sum_{i,j=1}^{2} X_{ij} r_i D^{-1} q_j \tag{3.151}$$

for $X = E_{\pm\alpha_1}$, i.e. define $\mathcal{M}_{E_{\alpha_1}} = r_1 D^{-1} q_2$ and $\mathcal{M}_{E_{-\alpha_1}} = r_2 D^{-1} q_1$.

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We are now in position to reformulate transformations (3.148) in one simple expression:

$$\delta^{(1)}_{\pm\alpha_1} \mathcal{L} = -\sum_{i=1}^{2} \delta^{(1)}_{\pm\alpha_1}(r_i) D^{-1} q_i - \sum_{i=1}^{2} r_i D^{-1} \delta^{(1)}_{\pm\alpha_1}(q_i) \equiv [\mathcal{M}_{E_{\pm\alpha_1}} \, , \, \mathcal{L}] \tag{3.152}$$

In calculating the left hand side of (3.152) we made use of identity:

$$f_1 D^{-1} g_1 f_2 D^{-1} g_2 = f_1 \partial^{-1}(g_1 f_2) D^{-1} g_2 - f_1 D^{-1} g_2 \partial^{-1}(g_1 f_2) \tag{3.153}$$

By letting X in eq. (3.151) to be σ_3 and introducing higher grade counterparts $\mathcal{L}^n(r_i)$, $(\mathcal{L}^*)^n(q_i)$ of r_i, q_i we can extend the above results to obtain the graded Borel loop algebra of $sl(2)$ within the pseudo-differential formalism. See reference [7] for details of this construction.

4 Additional Virasoro Symmetries

4.1 Virasoro Symmetry, the General Case

We consider first the general case of the constrained KP models described by the Lax operator $L = D_x + E + A$ within the $\hat{s}l(K + M + 1)$ algebra decomposed according to the grading operator $Q_{\mathbf{s}}$ from Section [6]. The semisimple element E of unit grade is given by (6.5) while the potential A is parametrized according to equation (6.13).

Define the modified "bare" Virasoro operators as

$$X_{m(K+1)} = (K+1) l_m - \sum_{j=M+1}^{M+K} \mu_j \cdot H^{(m)} \tag{4.1}$$

where μ_a are fundamental weights of $sl(M + K + 1)$ (as in Section [6]). The operators $l_m = -\lambda^m d = -\lambda^{m+1} d/d\lambda$ satisfy the centerless Virasoro algebra (4.2) :

$$[\, l_m \, , \, l_n \,] = (m - n) l_{m+n} \tag{4.2}$$

For b_N from $\mathcal{C}(\mathcal{K})$ defined in (6.9) and X_N from eq. (4.1) we find :

$$[\, X_{N'} \, , \, b_N \,] = -N b_{N+N'} \tag{4.3}$$

for $N' = n(K+1)$. These relations imply that the modified Virasoro generators $\tilde{X}_{N'}$ defined as :

$$\tilde{X}_{m(K+1)} \equiv X_{m(K+1)} - \sum_I t_I b_{I+m(K+1)} \tag{4.4}$$

satisfy the centerless Virasoro algebra (4.2) with indices which are multiples of $K + 1$

$$\left[\, \tilde{X}_{m(K+1)} \, , \, \tilde{X}_{n(K+1)} \,\right] = (m - n)(K + 1) \tilde{X}_{(m+n)(K+1)} \tag{4.5}$$

Following equation (3.8) we define now the symmetry transformations generated by the modified Virasoro generators $\tilde{X}_m$ as :

$$\delta^V_m A = [D_x + E + A \, , \, (\tilde{X}^{\Theta}_m)_+] \tag{4.6}$$

This generates the Borel-Virasoro algebra which is also a symmetry of the model due to the fact that it commutes with the isospectral flows :

$$\left(\delta^V_m \frac{d}{dt_n} - \frac{d}{dt_n}\delta^V_m\right)\Theta = 0 \quad , \quad m,n \geq 0 \tag{4.7}$$

The presence of the additional terms containing the time parameters t_I in definition (4.4) was crucial for commutativity with isospectral times established in (4.7).

4.2 The Homogeneous Gradation

We now turn our attention to the additional Virasoro symmetry in case of homogeneous gradation. Consider first the "bare" Virasoro operators $X_m = l_m = -\lambda^m d$, $m \geq 0$. in (3.4) which satisfy the Witt algebra (4.2).

In that case the relation (3.2) no longer holds. Instead one finds

$$[\, l_m \, , \, D_x + E \,] = -E^{(m+1)} \tag{4.8}$$

as a special case of $[\, l_m \, , \, b_n \,] = -n b_{m+n}$. Relation (4.8) can be rewritten as

$$\left[\, l_m - xE^{(m+1)} \, , \, D_x + E \,\right] = 0 \tag{4.9}$$

Applying Ad_Θ on (4.9) one finds the resolvent equation:

$$\left[\, \Theta\left(l_m - xE^{(m+1)}\right)\Theta^{-1} \, , \, \Theta(D_x+E)\Theta^{-1} \,\right] = 0 \tag{4.10}$$

since $l_m - xE^{(m+1)} = \exp(-xE)\, l_m \exp(xE)$.

Define, now

$$L_m = l_m - \sum_{i=1}^{\infty} i t_i E^{(m+i)} \;\; ; \;\; \text{for} \quad m \geq 0 \tag{4.11}$$

We are lead to:

Definition 4.1 *Define a transformation δ^V_m generated by L_m from eq.(4.11) as follows*

$$\delta^V_m \Theta \equiv (L^\Theta_m)_- \Theta \quad , \quad m \geq 0 \tag{4.12}$$

where we defined

$$L^\Theta_m \equiv \Theta L_m \Theta^{-1} = \Theta\left(l_m - \sum_{i=1}^{\infty} i t_i E^{(m+i)}\right)\Theta^{-1} \tag{4.13}$$

The Witt algebra of the "bare" generators L_m and L_m results via relation (3.25) for the Borel subalgebra of the Virasoro algebra

$$\left(\delta^V_m \delta^V_n - \delta^V_n \delta^V_m\right)\Theta = (m-n)\delta^V_{m+n}\Theta \quad , \quad m,n \geq 0 \tag{4.14}$$

4.3 Example: Virasoro Symmetry of AKNS (Sl (2)) Hierarchy

We now find action of the Virasoro symmetry on the Lax coefficients r, q from eq. (3.98) describing the AKNS hierarchy and compare with similar expressions found in the formalism based on the pseudo-differential operators [6].

Virasoro transformations of q, r are determined from:

$$\delta^V_n A = \left[\, L \, , \, (L^\Theta_n)_+ \,\right] = \left[\, D + A \, , \, (L^\Theta_n)_0 \,\right] = -\left[\, E \, , \, (L^\Theta_n)_{-1} \,\right] \tag{4.15}$$

for L_n from eq. (4.11). For the case of $\mathcal{G} = sl(2)$, $L^\Theta_m = \Theta L_m \Theta^{-1}$ can be expressed as:

$$L^\Theta_n = U\left(l_n - \sum_{j\geq 1} j\mathfrak{s}^{(-j)}\sigma_3 \lambda^{n-j} - \sum_{k\geq 1} k t_k E^{(k+n)}\right) U^{-1} \tag{4.16}$$

with $\mathfrak{s}^{(-1)} = -\partial^{-1}(qr), \mathfrak{s}^{(-2)} = \frac{1}{2}\partial_2 \ln\tau = \frac{1}{2}\partial^{-1}(rq_x - r_x q), \ldots$. Recall also, that $E^{(k)} = b_k = \lambda^k \sigma_3/2$.

We now proceed by calculating $\delta^V_n A$ from (4.15) for $n = 0, 1, 2$.

$\mathbf{n=0}$. We find from (4.16) that

$$\begin{aligned} (L^\Theta_0)_0 &= -d - \sum_{k\geq 1} k t_k (b^U_k)_0 && (4.17) \\ (L^\Theta_0)_{-1} &= -u^{(-1)}\lambda^{-1} - \mathfrak{s}^{(-1)}\lambda^{-1} - \sum_{k\geq 1} k t_k (b^U_k)_{-1} && (4.18) \end{aligned}$$

Plugging these two expressions into, respectively, (4.15) we find :

$$\delta^V_0 A = -A - \sum_{k\geq 1} k t_k \frac{dA}{dt_k} \tag{4.19}$$

or

$$\delta^V_0 r = -r - \sum_{k\geq 1} k t_k \frac{dr}{dt_k} \;\; ; \;\; \delta^V_0 q = -q - \sum_{k\geq 1} k t_k \frac{dq}{dt_k} \tag{4.20}$$

$\mathbf{n=1}$. We find from (4.16) that

$$\begin{aligned} (L^\Theta_1)_0 &= -(u^{(-1)} + \mathfrak{s}^{(-1)}\sigma_3) - \sum_{k\geq 1} k t_k (b^U_{k+1})_0 && (4.21) \\ (L^\Theta_1)_{-1} &= -2\lambda^{-1}(u^{(-2)} + \mathfrak{s}^{(-2)}\sigma_3) - \lambda^{-1}\left[\, u^{(-1)} \, , \, \mathfrak{s}^{(-1)}\sigma_3 \,\right] && (4.22) \\ &- \sum_{k\geq 1} k t_k (b^U_{k+1})_{-1} \end{aligned}$$

which lead via (4.15) to :

$$\begin{aligned} \delta^V_1 r &= -2r_x - 2r(\ln\tau)_x - \sum_{k\geq 1} k t_k \frac{dr}{dt_{k+1}} && (4.23) \\ \delta^V_1 q &= 2q_x + 2q(\ln\tau)_x - \sum_{k\geq 1} k t_k \frac{dq}{dt_{k+1}} && (4.24) \end{aligned}$$

$\mathbf{n=2}$. This time we find from (4.16) that

$$
\begin{aligned}
(L_2^{\Theta})_0 &= -2(u^{(-2)} + \mathfrak{s}^{(-2)}\sigma_3) - \left[u^{(-1)} \,,\, \mathfrak{s}^{(-1)}\sigma_3 \right] - \sum_{k\geq 1} kt_k (b^U_{k+2})_0 \\
(L_2^{\Theta})_{-1}|_{\mathcal{M}} &= -3\lambda^{-1}u^{(-3)} - \lambda^{-1}\left[u^{(-2)} \,,\, \mathfrak{s}^{(-1)}\sigma_3 \right] \\
&- 2\lambda^{-1}\left[u^{(-1)} \,,\, \mathfrak{s}^{(-2)}\sigma_3 \right] - \sum_{k\geq 1} kt_k (b^U_{k+2})_{-1}|_{\mathcal{M}}
\end{aligned}
\tag{4.25}
$$

Plugging expression from (4.25) into (4.15) we obtain:

$$
\delta_2^V r = -3r_{xx} - 2r_x(\ln\tau)_x - 2r\partial_2(\ln\tau) + 4qr^2 - \sum_{k\geq 1} kt_k \frac{dr}{dt_{k+2}} \tag{4.26}
$$

$$
\delta_2^V q = -3q_{xx} - 2q_x(\ln\tau)_x + 2q\partial_2(\ln\tau) + 4q^2 r - \sum_{k\geq 1} kt_k \frac{dq}{dt_{k+2}} \tag{4.27}
$$

The crucial observation is that the transformation:

$$
\delta_n^V \;\rightarrow\; \tilde{\delta}_n^V \equiv \delta_n^V + \frac{(n+1)}{2}\frac{d}{dt_n} \tag{4.28}
$$

preserves the Virasoro algebra, meaning that $\tilde{\delta}_n^V$ satisfies the Virasoro algebra. Taking into account that d/dt_n is generated by B_n one obtains the following expressions:

$$
\tilde{\delta}_0^V r = -r/2 - \sum_{k\geq 1} kt_k \frac{dr}{dt_k} \tag{4.29}
$$

$$
\tilde{\delta}_1^V r = -r_x - 2r(\ln\tau)_x - \sum_{k\geq 1} kt_k \frac{dr}{dt_{k+1}} \tag{4.30}
$$

$$
\tilde{\delta}_2^V r = -\frac{3}{2}r_{xx} - 2r_x(\ln\tau)_x - 2r\partial_2(\ln\tau) + qr^2 - \sum_{k\geq 1} kt_k \frac{dr}{dt_{k+2}} \tag{4.31}
$$

We will now attempt to rewrite the above relations in the Sato pseudo-differeential Lax formalism. For this purpose we need to introduce Orlov-Shulman operator M in addition to the Lax operator $\mathcal{L} = D - rD^{-1}q = D + \Phi D^{-1}\Psi$. M is defined in such a way that

$$
M\psi_{BA}(t,\lambda) = \frac{\partial}{\partial\lambda}\psi_{BA}(t,\lambda) \tag{4.32}
$$

for the Baker-Akhiezer wave function :

$$
\ln\psi_{BA}(t,\lambda) = \sum_{n=1}^{\infty} t_n\lambda^n + \sum_{n=1}^{\infty} \lambda^{-n} p_n(-[\partial])\ln\tau \tag{4.33}
$$

and therefore Orlov-Shulman operator M can be written as

$$
M = \sum_{n=1}^{\infty} nt_n \mathcal{L}^{n-1} + \sum_{n=1}^{\infty} \left(-np_n(-[\partial])\ln\tau\right)\, \mathcal{L}^{-n-1} \tag{4.34}
$$

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Note, that since $\mathcal{L}\psi_{BA}(t,\lambda) = \lambda\psi_{BA}(t,\lambda)$ we have $[\mathcal{L}\,,\, M] = 1$

Using representation of the Orlov-Shulman operator given above in equation (4.34) and identity $\partial_n\partial_x \ln\tau = -\mathrm{Res}(\mathcal{L}^n)$ for $n = 1, 2$ we can rewrite relations (4.29)-(4.31) as :

$$
\tilde{\delta}_0^V \Phi = -\Phi/2 - (M\mathcal{L})_+(\Phi) \tag{4.35}
$$

$$
\tilde{\delta}_1^V \Phi = -\mathcal{L}(\Phi) - \left(M\mathcal{L}^2\right)_+(\Phi) \tag{4.36}
$$

$$
\tilde{\delta}_2^V \Phi = -\frac{3}{2}\mathcal{L}^2(\Phi) - X_2^{(1)}(\Phi) - (M\mathcal{L}^3)_+(\Phi) \tag{4.37}
$$

where use was made of identification of r, q with $\Phi, -\Psi$ (see f.i. (3.115)) and where the pseudo-differential object $X_2^{(1)}$

$$
X_2^{(1)} = -\frac{1}{2}\mathcal{L}(\Phi)D^{-1}\Psi + \frac{1}{2}\Phi D^{-1}\mathcal{L}^*(\Psi) \tag{4.38}
$$

is a special case of

$$
X_k^{(1)} = \sum_{j=0}^{k-1} [j - \frac{1}{2}(k-1)]\mathcal{L}^{k-1-j}(\Phi)D^{-1}(\mathcal{L}^*)^j(\Psi) \tag{4.39}
$$

All these results in (4.35)-(4.37) agree perfectly well with reference [6] (up to an overall minus sign).

Now we deal with action of Virasoro transformations $\tilde{\delta}_n^V \equiv \delta_n^V + \frac{(n+1)}{2}\frac{d}{dt_n}$ from (4.28) applied on q.

Recalling that q is an adjoint eigenfunction i.e. $\partial q/dt_n = -B_n^*(q)$ we obtain the following expressions:

$$
\tilde{\delta}_0^V q = -3q/2 - \sum_{k\geq 1} kt_k \frac{dq}{dt_k} \tag{4.40}
$$

$$
\tilde{\delta}_1^V q = 3q_x + 2q(\ln\tau)_x - \sum_{k\geq 1} kt_k \frac{dq}{dt_{k+1}} \tag{4.41}
$$

$$
\tilde{\delta}_2^V q = -\frac{9}{2}q_{xx} - 2q_x(\ln\tau)_x + 2q\partial_2(\ln\tau) + 7q^2 r - \sum_{k\geq 1} kt_k \frac{dq}{dt_{k+2}} \tag{4.42}
$$

Since

$$
\ln\psi^*_{BA}(t,\lambda) = -\sum_{n=1}^{\infty} t_n\lambda^n + \sum_{n=1}^{\infty} \lambda^{-n} p_n([\partial])\ln\tau \tag{4.43}
$$

we find that M^* such that:

$$
M^*\psi^*_{BA}(t,\lambda) = -\frac{d}{d\lambda}\psi^*_{BA}(t,\lambda) \tag{4.44}
$$

is equal :

$$
M^* = \sum_{n=1}^{\infty} nt_n (\mathcal{L}^*)^{n-1} + \sum_{n=1}^{\infty} \left(np_n([\partial])\ln\tau\right)\, (\mathcal{L}^*)^{-n-1} \tag{4.45}
$$

and satisfies $[M^*, \mathcal{L}^*] = 1$.

With this definition and identification (3.115) relations (4.40)-(4.42) take form

$$\tilde{\delta}_0^V \Psi = -\Psi/2 + (M\mathcal{L})_+^* (\Psi) \qquad (4.46)$$

$$\tilde{\delta}_1^V \Psi = -\mathcal{L}^*(\Psi) + \left(M\mathcal{L}^2\right)_+^* (\Psi) \qquad (4.47)$$

$$\tilde{\delta}_2^V \Psi = -\frac{3}{2}(\mathcal{L}^*)^2(\Psi) + X_2^{(1)*}(\Psi) + (M\mathcal{L}^3)_+^*(\Psi) \qquad (4.48)$$

with

$$X_2^{(1)*} = \frac{1}{2}\Psi D^{-1}\mathcal{L}(\Phi) - \frac{1}{2}\mathcal{L}^*(\Psi)D^{-1}\Phi \qquad (4.49)$$

Relations (4.46)-(4.48) again agree with the reference [6] (up to an overall minus sign).

5 Fermionic Symmetry Flows from the Super Algebra

Consider the super-algebra $A(p, s)$ composed of the bosonic sub-algebra

$$SL(p+1) \otimes SL(s+1) \otimes U(1) \qquad (5.1)$$

together with the fermionic generators $E_{\pm(\alpha_i+\alpha_{i+1}+\cdots+\alpha_{p+1}+\alpha_{p+2}+\cdots+\alpha_j)}$, $i = 1, \cdots, p+1$, $j = p+1, \cdots, p+s+1$.

The root system can be realized in terms of $p+s+1$ orthonormal vectors, $e_i \cdot e_k = \delta_{ik}$, $f_j \cdot f_l = -\delta_{jl}$, $i, k = 1, \cdots, p+1$, $j, l = 1, \cdots, s+1$. Let $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \cdots, \alpha_p = e_p - e_{p+1}$ and $\alpha_{p+2} = f_1 - f_2, \cdots, \alpha_{p+s+1} = f_s - f_{s+1}$ be the bosonic simple roots of $SL(p+1) \otimes SL(s+1)$ and denote by $\alpha_{p+1} = e_{p+1} - f_1$ the fermionic simple root.

We now define $E = h_{p+1}^{(1)} = \alpha_{p+1} \cdot H^{(1)}$ as the constant Lie algebra valued element with unit grade with respect to the homogeneous grading $Q = d$ which defines decomposition of the super Lie algebra $A(p, s)$ on kernel and image of ad E. The grade zero part of the kernel is :

$$\begin{aligned} \text{Ker (ad } E)_0 \;=\; & \{SL(p) \otimes SL(s) \otimes U(1)^3\} \oplus \{E_{\pm\alpha_{p+1}}^{(0)}, E_{\pm(\alpha_p+\alpha_{p+1}+\alpha_{p+2})}^{(0)}, \\ & \ldots, E_{\pm(\alpha_i+\alpha_{i+1}+\cdots+\alpha_{p+1}+\cdots+\alpha_{p+j})}\} \end{aligned} \qquad (5.2)$$

where $i = 1, \ldots, p+1$, $j = 1, \ldots, s+1$ and $U(1)^3$ is generated by h_{p+1} and $\mu_p \cdot H, \mu_{p+2} \cdot H$. The presence of the fermionic generators

$$E_{\pm(\alpha_i+\cdots+\alpha_{p+1}+\cdots+\alpha_{p+j})}$$

in Ker (ad E) is a consequence of the indefinite metric as expressed by $e_i \cdot e_k = \delta_{ik}$, $f_k \cdot f_l = -\delta_{kl}$.

The center of Ker (ad E) is generated by h_{p+1} and is related with the bosonic flows.

In addition, we introduce fermionic elements $F_\pm \equiv E_{\alpha_{p+1}}^{(0)} \pm E_{-\alpha_{p+1}}^{(1)}$ from eq.(5.2) whose squares reproduce the unit grade constant element according to $\frac{1}{2}\{F_\pm, F_\pm\} = \pm h_{p+1}^{(1)} = \pm E$. Moreover it holds that $\{F_\pm, F_\mp\} = 0$. We note that according to gradation $Q' = 2d + \mu_{p+1} \cdot H$ the elements $F_\pm$ possess the unit grade.

We can generalize these definitions to fermionic elements of grade n with respect to grading defined by Q' as $F_\pm^{[n]} \equiv E_{\alpha_{p+1}}^{(n)} \pm E_{-\alpha_{p+1}}^{(n+1)}$, $n \geq 0$. They satisfy the anti-commutation relations $\frac{1}{2}\{F_\pm^{[n]}, F_\pm^{[m]}\} = \pm h_{p+1}^{(m+n+1)}$ and $\{F_\pm^{[n]}, F_\mp^{[m]}\} = 0$.

We can now associate symmetry flows to the elements $F_\pm^{[n]}$ within our dressing framework. This is illustrated in the following example.

5.1 Example of Super-algebra sl(2|1)

The super-algebra $sl(2|1)$ is the (N=2) extended supersymmetric version of $sl(2)$ and contains four bosonic generators $E_{\alpha_1}, E_{-\alpha_1}, H_1, H_2$ which form the Lie algebra $sl(2) \oplus U(1)$ and four fermionic generators $E_{\alpha_2}, E_{-\alpha_2}, E_{\alpha_1+\alpha_2}$ and $E_{-\alpha_1-\alpha_2}$. Here $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2 - f_1$ denote simple bosonic and fermionic roots. In this notation we have the following Cartan elements

$$\alpha_2 \cdot H = H_1 + H_2 \quad ; \quad -(\alpha_1 + \alpha_2) \cdot H = H_1 - H_2 \quad ; \quad -\alpha_1 \cdot H = 2H_1 \qquad (5.3)$$

The three-dimensional matrix representation (fundamental representation) of the above operators in the Cartan-Weyl basis reads as

$$H_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad H_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_{-\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_{\alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_{\alpha_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_{-\alpha_1-\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$E_{-\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad E_{\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The model is considered with the homogeneous gradation. Furthermore, a semisimple grade-one element is taken to be $E = \lambda\alpha_2 \cdot H = \lambda(H_1 + H_2) = \lambda \text{diag}(1, 0, 1)$. Accordingly, the corresponding kernel $\mathcal{K}$

$$\mathcal{K} = \text{Ker (ad } E) = \{\lambda^n H_1, \lambda^n H_2, \lambda^n E_{\alpha_2}, \lambda^n E_{-\alpha_2}\} \qquad (5.4)$$

contains fermionic roots. We consider an element $F = E_{\alpha_2}^{(0)} + E_{-\alpha_2}^{(1)}$ such that $F^2 = E$. The role of F was recognized already in [18] in construction of the fermionic Lax operator for s-AKNS hierarchy. Note, that according to gradation $Q' = 2d + H_1 - H_2$ the element F possesses a unit grade.

The higher grade generalizations of F defined as $F_\pm^{[n]} \equiv E_{\alpha_2}^{(n)} \pm E_{-\alpha_2}^{(n+1)}$ for $n \geq 0$ are in $\mathcal{K}$ and have grade $2n + 1$ with respect to grading defined by Q'. They satisfy the commutation relations

$$\{F_\pm^{[n]}, F_\pm^{[m]}\} = \pm 2(H_1 + H_2)^{(m+n+1)} = \pm 2E^{(m+n+1)} \qquad (5.5)$$

and

$$\{F_\pm^{[n]}, F_\mp^{[m]}\} = 0 \qquad (5.6)$$

In addition, we will also list commutation relations of $F_\pm^{[n]}$ with another (apart from E) Cartan operator $H_1 - H_2$ from $\mathcal{K}$:

$$\{(H_1 - H_2)^{(n)}\,,\, F_\pm^{[m]}\} = -F_\mp^{[n+m]} \tag{5.7}$$

We encounter here an example of the model which contains fermionic elements in Ker (ad E). Accordingly, the above algebraic structure will give rise to the graded algebra of flows as follows. Let the bosonic Lax $L = D + E + A$ be given with the potential A, as usually, determined by the condition that all its components are in the zero-grade subspace of $\mathcal{M} =$ Im (ad E) [13] :

$$A = b_1 E_{-\alpha_1} + b_2 E_{\alpha_1} + f_1 E_{\alpha_1+\alpha_2} + f_2 E_{-\alpha_1-\alpha_2} = \begin{pmatrix} 0 & b_1 & 0 \\ b_2 & 0 & f_1 \\ 0 & f_2 & 0 \end{pmatrix} \tag{5.8}$$

Next, we associate the symmetry flows $\partial/\partial\tau_n^\pm$ to the the odd elements $F_\pm^{[n]}$ according to the definition:

$$\frac{\partial}{\partial\tau_n^\pm}\Theta \;=\; \delta_{F_\pm}^{(n)}\Theta \;=\; (\Theta F_\pm^{[n]}\Theta^{-1})_-\Theta \tag{5.9}$$

which furthermore are assumed to anti-commute with fermionic roots and so

$$\frac{\partial}{\partial\tau_n^\pm}F_\pm^{[m]} = -F_\pm^{[m]}\frac{\partial}{\partial\tau_n^\pm} \tag{5.10}$$

Next, we associate the symmetry flows $\partial/\partial u_n$ to Cartan operator $(H_1 - H_2)^{(n)}$ via:

$$\frac{\partial}{\partial u_n}\Theta \;=\; (\Theta(H_1 - H_2)^{(n)}\Theta^{-1})_-\Theta \tag{5.11}$$

From equations (3.25) and (3.28) and eq.(5.5) we find that the fermionic flows commute with isospectral flows and close into the isospectral flows generated by $E^{(n)}$ as follows

$$\left(\frac{\partial}{\partial\tau_m^\pm}\frac{\partial}{\partial\tau_n^\pm} + \frac{\partial}{\partial\tau_n^\pm}\frac{\partial}{\partial\tau_m^\pm}\right)\Theta = \pm 2\frac{\partial}{\partial t_{m+n+1}}\Theta \quad , \quad m,n \geq 0 \tag{5.12}$$

and satisfy in addition

$$\left(\frac{\partial}{\partial u_m}\frac{\partial}{\partial\tau_n^\pm} - \frac{\partial}{\partial\tau_n^\pm}\frac{\partial}{\partial u_m}\right)\Theta = -\frac{\partial}{\partial\tau_{m+n}^\mp}\Theta \quad , \quad m,n \geq 0 \tag{5.13}$$

Notice that algebra of the flows $\partial/\partial\tau_n^-$ is isomorphic to the Manin-Radul algebra of flows. The extended algebra of flows $\partial/\partial\tau_n^\pm$, $\frac{\partial}{\partial u_m}$ together with isospectral flows has been encountered in the study of the maximal SKP hierarchy (see e.g. [19], [20]).

We have shown above that the model possesses additional fermionic symmetry flows. Let us now find their explicit form. Via the dressing technique we arrive at

$$\delta_F^{(1)}A \equiv [L\,,\,(\Theta F\Theta^{-1})_+] = [\partial_x + A\,,\,(\Theta F\Theta^{-1})_0] \tag{5.14}$$

Working out the lowest terms $u^{(-1)}, \mathfrak{s}^{(-1)}$ in the grading expansion of Θ and plugging them into expression for $(\Theta F\Theta^{-1})_0$ in (5.14) we obtain for b_1 and f_1 components of A in (5.8) the following transformations :

$$\delta_F^{(1)}b_1 = -f_2 + b_1\int b_1 f_1 \quad ; \quad \delta_F^{(1)}f_1 = b_2 + f_1\int b_1 f_1 \tag{5.15}$$

To understand these flows and their connection to supersymmetry we recall from [13] that the matrix spectral problem $L\Psi = 0$ with $L = D + E + A$ can be reformulated in the equivalent form of the scalar spectral problem : $\mathcal{L}\psi_{BA} = \lambda\psi_{BA}$ with the pseudo-differential Lax operator:

$$\mathcal{L} = D + \Phi(t,\theta){D_\theta}^{-1}\Psi(t,\theta) \tag{5.16}$$

where the superfields $\Phi(t,\theta)$ and $\Psi(t,\theta)$ are, respectively, eigenfunctions and adjoint eigenfunctions of $\mathcal{L}$ and D_θ is a covariant derivative of the form: $D_\theta = \frac{\partial}{\partial\theta} + \theta\partial$, which satisfies ${D_\theta}^2 = \partial$.

The paper [13] established the following connection between components of A and the superfields $\Phi(t,\theta)$ and $\Psi(t,\theta)$:

$$b_1 = \Phi(t,\theta)\;,\; f_1 = \Psi(t,\theta)\;,\; b_2 = -D_\theta\Psi + (\int\Phi\Psi)\Psi\;,\; f_2 = D_\theta\Phi + (\int\Phi\Psi)\Phi \tag{5.17}$$

Inserting these values into the transformation law (5.15) we find that

$$\delta_F^{(1)}\Phi(t,\theta) = -D_\theta\Phi(t,\theta) \quad , \quad \delta_F^{(1)}\Psi(t,\theta) = -D_\theta\Psi(t,\theta) \tag{5.18}$$

Hence the first flows associated to F amount to application of the covariant derivative. In order to find the higher flows $\delta_F^{(2n+1)}$ generated by $F_+^{[n]}$ we employ the recursion techniques from [13] generalized to odd/half-integer flows $\partial/\partial t_{2n+1} \equiv \delta_F^{(2n+1)}$ entering the zero curvature equation:

$$\frac{\partial}{\partial t_{2n+1}}A - \partial B_{2n+1} + \lambda[E\,,\,B_{2n+1}] + [A\,,\,B_{2n+1}] = 0 \tag{5.19}$$

with

$$B_{2n+1} = F_+^{[n]} + B_n + \ldots + B_0 \tag{5.20}$$

where terms B_k have grade equal to k. After plugging expansion (5.20) into relation (5.19) and decomposing it according to the grade we find

$$\delta_F^{(2n+1)}(A_E) = (-\mathcal{R})^n\left(\delta_F^{(1)}(A_E)\right) \tag{5.21}$$

where $A_E \equiv ad_E(A)$ and the recursion matrix is given by :

$$\mathcal{R} \equiv ad_E\left(\partial - ad_{\mathcal{A}}\,\partial^{-1}ad_{\mathcal{A}}\right) \tag{5.22}$$

6 Background on Graded Affine Lie Algebras

In this section we provide the basic ingredients about the graded affine Lie algebras needed in construction of integrable hierarchies of the constrained KP type, for more details see [5] and references therein.

Let $\widehat{\mathcal{G}}$ be an affine Lie algebra, and $\mathcal{G}$ be the finite dimensional simple Lie algebra associated to it. The integral gradation of $\widehat{\mathcal{G}}$ defines the following decomposition :

$$\widehat{\mathcal{G}} = \bigoplus_{n\in\mathbb{Z}} \widehat{\mathcal{G}}_n(\mathbf{s})\,, \qquad [\widehat{\mathcal{G}}_m(\mathbf{s})\,,\,\widehat{\mathcal{G}}_n(\mathbf{s})] \subset \widehat{\mathcal{G}}_{m+n}(\mathbf{s}) \tag{6.1}$$

where $\widehat{\mathcal{G}}_n(\mathbf{s})$ is a grade-n subspace:

$$[Q_{\mathbf{s}}\,,\,\widehat{\mathcal{G}}_n(\mathbf{s})] = n\,\widehat{\mathcal{G}}_n(\mathbf{s}) \tag{6.2}$$

with respect to the grading operator :

$$Q_{\mathbf{s}} \equiv \sum_{a=1}^{r} s_a \,\frac{2\mu_a\cdot H^0}{\alpha_a^2} + N_{\mathbf{s}} d \tag{6.3}$$

The following ingredients entered the definition (6.3). The vector $\mathbf{s} = (s_0, s_1, \ldots, s_r)$ [21], has components s_i being non negative relatively prime integers, and $r \equiv \operatorname{rank}\mathcal{G}$. Moreover, H_a^0, $a = 1,2,\ldots,r$, are the Cartan sub-algebra generators of $\mathcal{G}$, μ_a its fundamental weights satisfying $\frac{2\mu_a\cdot\alpha_b}{\alpha_b^2} = 2\delta_{ab}$, with α_a being the simple roots of $\mathcal{G}$. $d = \lambda d/d\lambda$ is the usual derivation of $\widehat{\mathcal{G}}$, responsible for the homogeneous gradation of $\widehat{\mathcal{G}}$, corresponding to $\mathbf{s}_{\mathrm{hom}} = (1,0,0,\ldots,0)$. In addition, we have, $N_{\mathbf{s}} \equiv \sum_{i=0}^{r} s_i m_i^{\psi}$, $\psi = \sum_{a=1}^{r} m_a^{\psi}\alpha_a$, $m_0^{\psi} = 1$, where ψ is the highest positive root of $\mathcal{G}$.

6.1 The case of $\widehat{\mathcal{G}} = \widehat{\mathbf{sl}}(\mathbf{M}+\mathbf{K}+\mathbf{1})$

We now apply the above formalism to the example of the affine Lie algebra $\widehat{\mathcal{G}} = \widehat{sl}\,(M+K+1)$, $(A^{(1)}_{M+K})$ furnished with gradation $\mathbf{s}$ and corresponding grading operator $Q_{\mathbf{s}}$:

$$\mathbf{s} = (1, \underbrace{0,\ldots,0}_{M}, \underbrace{1,\ldots,1}_{K}) \;\; ; \;\; Q_{\mathbf{s}} = \sum_{j=M+1}^{M+K} \mu_j\cdot H^{(0)} + (K+1)d \tag{6.4}$$

We will denote the simple roots of $\widehat{sl}\,(M+K+1)$ by α_j, $j = 0,1,\ldots,M+K$, with $\alpha_0 \equiv -\psi$ for ψ being the highest positive root of $\mathcal{G} = sl\,(M+K+1)$. All roots are such that $\alpha_j^2 = 2$.

The semisimple, grade-one (w.r.t. to gradation $\mathbf{s}$) element E is taken to be :

$$E = \sum_{j=M+1}^{M+K} E^{(0)}_{\alpha_j} + E^{(1)}_{-(\alpha_{M+1}+\cdots+\alpha_{M+K})} \tag{6.5}$$

it's centralizer is :

$$\mathcal{K} = \mathrm{Ker}\;(\mathrm{ad}\;E) = \{\hat{K}_0 \equiv \widehat{sl}(M)\oplus\hat{U}(1)\;,\;\hat{\mathcal{H}}_K\} \tag{6.6}$$

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where $\widehat{sl}(M)$ is the affine Lie sub-algebra of $\widehat{\mathcal{G}} = \widehat{sl}(M+K+1)$ with simple roots α_j, $j = 1,2,\ldots,M-1$ and $\alpha_0 = -(\alpha_1+\alpha_2+\ldots+\alpha_{M-1})$. The algebra $\hat{U}(1)$ is generated by $\mu_M\cdot H^{(k)}$, $k\in\mathbb{Z}$. In addition, $\hat{\mathcal{H}}_K$ is the sub-algebra of $\widehat{sl}(K+1) \in \widehat{sl}(M+K+1)$ and spanned by generators :

$$\begin{aligned} E_{l+(K+1)n} = {} & E^{(n)}_{\alpha_{M+1}+\alpha_{M+2}+\ldots+\alpha_{M+l}} + E^{(n)}_{\alpha_{M+2}+\alpha_{M+3}+\ldots+\alpha_{M+l+1}} + \ldots \\ & + \; E^{(n)}_{\alpha_{M+K-l+1}+\alpha_{M+K-l+2}+\ldots+\alpha_{M+K-1}+\alpha_{M+K}} \\ & + \; E^{(n+1)}_{-(\alpha_{M+1}+\alpha_{M+2}+\ldots+\alpha_{M+K-l+1})} + E^{(n+1)}_{-(\alpha_{M+2}+\alpha_{M+3}+\ldots+\alpha_{M+K-l})} \\ & + \; \ldots + E^{(n+1)}_{-(\alpha_{M+l}+\alpha_{M+3}+\ldots+\alpha_{M+K})} \end{aligned} \tag{6.7}$$

with $l = 1,2,3,\ldots,K$. Note, that $E_1 = E$. These generators satisfy

$$\Big[\,Q_{\mathbf{s}}\,,\,E_{l+(K+1)n}\Big] = (l+(K+1)n)\,E_{l+(K+1)n} \tag{6.8}$$

Also, we have

$$\mathcal{C}(\mathcal{K}) = \text{center}\,\mathrm{Ker}\;(\mathrm{ad}\;E) = \{\hat{U}(1)\;,\;\hat{\mathcal{H}}_K\} \tag{6.9}$$

where $\hat{U}(1)$ is as in eq.(6.6). Notice that $\Big[\,Q_{\mathbf{s}}\,,\,\mu_M\cdot H^{(k)}\Big] = k(K+1)\mu_M\cdot H^{(k)}$. The center of Ker (ad E) has one and only one generator associated to a given grade according to the scheme:

$$\begin{aligned} b_N &= E_{N=l+(K+1)n} \qquad l = 1,2,\ldots,K \qquad &(6.10)\\ b_{k(K+1)} &= \mu_M\cdot H^{(k)}\,, \qquad k\in\mathbb{Z} \qquad &(6.11) \end{aligned}$$

According to (3.4), each of the generators from the center of Ker (ad E)in (6.10)-(6.11) will give rise to the corresponding isospectral flows with times $t_{b_N}, t_{b_{k(K+1)}}$. In particular the element $E_1 = E$ will generate the flow corresponding to $\partial/\partial t_1 = \partial/\partial x$.

The generators of the complement $\mathcal{M}$ of $\mathcal{K}$ within the grade zero sub-algebra $\widehat{\mathcal{G}}_0$ are :

$$\mathcal{M}_0 = \{P_{\pm i} = E^{(0)}_{\pm(\alpha_i+\alpha_{i+1}+\ldots+\alpha_M)}\,,\,\alpha_a\cdot H^{(0)}\} \tag{6.12}$$

for $i = 1,2,\ldots,M$ and $a = M+1,\ldots,M+K$. Accordingly, we parametrize the potential A, as follows

$$A_0 = \sum_{i=1}^{M}(q_i P_i + r_i P_{-i}) + \sum_{a=M+1}^{M+K} U_a\;\alpha_a\cdot H^{(0)} \tag{6.13}$$

where q_i, r_i and U_a are fields of the model.

6.1.1 The case $K = 0$

In this case, we have $\widehat{\mathcal{G}} = \widehat{sl}(M+1)$ and $Q_{\mathbf{s}} \equiv d$. The latter defines the homogeneous gradation. This example was discussed in detail in ref. [12]. The semisimple grade-one element E is here given by

$$E = \mu_M\cdot H^{(1)} \tag{6.14}$$

The kernel of ad E is :

$$\mathcal{K} = \mathrm{Ker}\;(\mathrm{ad}\;E) = \{\widehat{sl}(M)\oplus\hat{U}(1)\} \tag{6.15}$$

with $\hat{U}(1)$ being generated by $\mu_M \cdot H^{(k)}$, $k \in \mathbb{Z}$ and defining the center of Ker (ad E) :

$$\mathcal{C}(\mathcal{K}) = \text{center Ker (ad } E) = \{\mu_M \cdot H^{(k)}\ ,\ k \in \mathbb{Z}\} \tag{6.16}$$

Therefore, the dressing formalism associates the isospectral flow for each element :

$$b_k \equiv \mu_M \cdot H^{(k)}\,, \qquad k \text{ being a positive integer} \tag{6.17}$$

The potential A :

$$A = \sum_{i=1}^{M} \Big(q_i E^{(0)}_{(\alpha_i+\alpha_{i+1}+\ldots+\alpha_M)} + r_i E^{(0)}_{-(\alpha_i+\alpha_{i+1}+\ldots+\alpha_M)} \Big) \tag{6.18}$$

lies in the complement $\mathcal{M}$ of $\mathcal{K}$ within $\widehat{\mathcal{G}}_0$. Note, that $\widehat{\mathcal{G}}/\mathcal{K}$ is now a symmetric space.

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